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On the prevention or facilitation of buckling of beams

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ABSTRACT

The present paper considers the use of linear or rotary transverse springs to: (i) prevent the buckling of beams by raising the critical buckling load and thus allow a larger axial tension; (ii) provoke the buckling of beams, by lowering the critical buckling load, facilitating the demolition of a structure. The prevention or facilitation of buckling depends on the positioning of the linear and/or rotary springs to oppose or favour bending, for example: at the tip of a (i) cantilever or clamped–free beam; at the middle of a (ii) clamped–clamped, (iii) pinned–pinned or (iv) clamped–pinned beam. In all eight of the four beam supports (i)–(iv) with either linear or rotary springs, the relation between the critical buckling load and the resilience of the spring is obtained.

1. Introduction

The buckling of beams is the simplest elastic instability, first studied shortly after the theory of the elastica was established [1,2]. The linear theory of the buckling of beams assuming small slope of the elastica is a textbook subject [3]. The non-linear theory of the buckling of beams, allowing for large slope of the elastica, has been presented elsewhere [4]. The prevention of buckling, by raising the critical buckling load, is desirable to allow most structures to withstand a larger axial load. Conversely to demolish more easily an existing structure, the critical buckling load may be reduced so that a smaller axial load can lead to failure. Buckling together with the collapse and displacement and twist are four simple instabilities of elastic beams [5]. The displacement (twist) instabilities are caused by linear (rotary) springs placed transversely on one side of the beam and are prevented by placing the springs transversely on the other side of the beam.

The present paper addresses the prevention (provocation) of the buckling of beams using a linear (Section 2) or rotary (Section 3) spring placed transversely on one side so as to raise (Fig. 1) [lower (Fig. 2)] the critical axial buckling load. The spring is placed at the tip [Fig. 1a (Fig. 2a) and Section 2.1 (Section 3.1)] on one side to raise (lower) the critical buckling load for a (i) cantilever beam, that is with one clamped and one free end. The spring is placed at the mid position on one side to raise (lower) the critical buckling load for the other three classical supports, namely: (ii) clamped–clamped in Fig. 1b (Fig. 2b) and Section 2.2 (Section 3.2); (iii) pinned–pinned in Fig. 1c (Fig. 2c) and Section 2.3 (Section 3.3); (iv) clamped–pinned in Fig. 1d (Fig. 2d) and Section 2.4 (Section 3.4). In all eight cases combining the four supports (i)–(iv) with linear (Section 2) or rotary (Section 3) springs, the relation between the resilience of the spring and the critical buckling load is obtained. This relation is analysed for all values of the two parameters (Section 4) for all buckling modes (Section 4.1) both analytically(Section 4.2) and graphically (Sections 4.3 and 4.4), for a cantilever beam with a rotary (linear) spring at the tip [Table 2 (Table 3) and Figs. 5 and 6 (Figs. 7 and 8)]. The main result namely the resilience of the linear or rotary that (a) placed transversely on one side prevents buckling under a larger axial load and by raising the critical buckling load makes the structure safer; (b) placed transversely on the opposite side causes buckling by a smaller axial load by decreasing the critical buckling load, providing a simpler alternative to explosive demolition of structures.

2. Effect of a linear spring on buckling

The buckling of a beam may be shifted to a higher (lower) critical load by positioning a transverse spring so as to oppose (Fig. 1) [favour (Fig. 2)] bending. The spring is placed at the free end [Fig. 1a (Fig. 2a)] for a cantilever beam (Section 2.1), and at mid-position for the other classical combinations of support: (Section 2.2) clamped–clamped [Fig. 1b (Fig. 2b)]; (Section 2.3)

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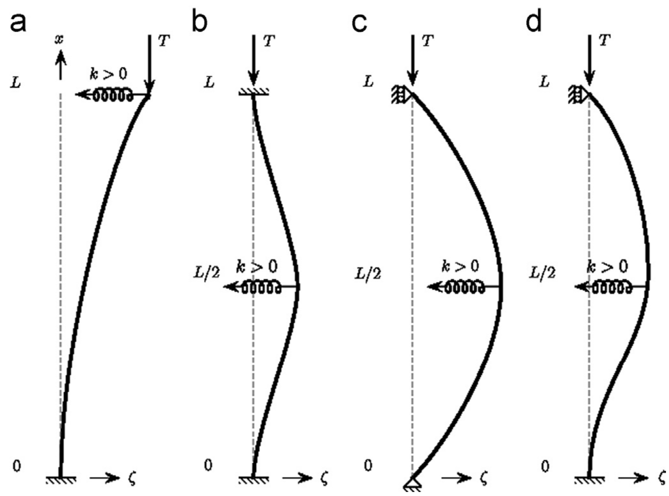

Fig. 1. Spring placement to raise the critical buckling load. (a) Cantilever, (b) clamped–clamped, (c) pinned–pinned, (d) clamped–pinned.

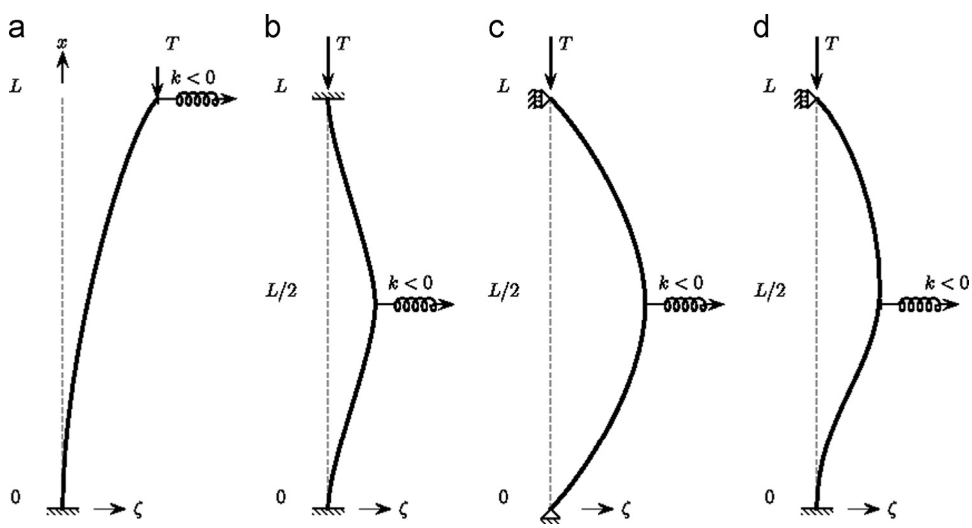

Fig. 2. Spring placement to lower the critical buckling load. (a) Cantilever, (b) clamped–clamped, (c) pinned–pinned, (d) clamped–pinned.

pinned–pinned [Fig. 1c (Fig. 2c)]; (Section 2.4) clamped–pinned [Fig. 1d (Fig. 2d)].

2.1. Cantilever beam with spring support

The linear equation of the elastica of a beam valid for small slope (1a) is (1b):

$$\zeta'^2 \ll 1: \quad EI\zeta'''' + T\zeta'' = 0, \tag{1a, b}$$

specifying $\zeta(x)$ the transverse displacement ζ as a function of the longitudinal coordinate x; the product of the Young modulus E of the material and moment of inertia I of the cross-section is the bending stiffness, assumed to be constant. The longitudinal tension T is a uniform compression, whose ratio to the bending stiffness (2a) is the only parameter in the equation of the elastica (1b) $\equiv$ (2b):

$$p = \sqrt{\frac{T}{EI}} = \text{const}: \quad \zeta'''' + p^2\zeta'' = 0. \tag{2a, b}$$

The general integral of (2b) is

$$\zeta(x) = A + Bx + C\cos(px) + D\sin(px), \tag{3}$$

where (A, B, C, D) are arbitrary constants. The clamping conditions at one end $x=0$:

$$0 = \zeta(0) = A + C, \quad 0 = \zeta'(0) = B + Dp, \tag{4a, b}$$

eliminate two of the constants of integration in (3) leading to

$$\zeta(x) = A[1 - \cos(px)] + D[\sin(px) - px], \tag{5}$$

for the shape of the elastica.

The boundary conditions at the free end state the vanishing of the bending moment (6a) and that the transverse force is balanced by the force of the spring (6b):

$$\zeta''(L) = 0, \quad EI\zeta'''(L) + T\zeta'(L) = k\zeta(L), \tag{6a, b}$$

where $k>0$ $(k<0)$ if the spring opposes (favours) buckling, that may be expected to increase (decrease) the critical buckling load for instability. The boundary condition (6b) $\equiv$ (7b) involves two parameters:

$$q = \frac{k}{EI}: \quad \zeta'''(L) + p^2\zeta'(L) = q\zeta(L), \tag{7a, b}$$

namely (2a) and the ratio of the resilience of the spring to the bending stiffness (7a). Substituting the shape of the elastica in the boundary conditions (6a) and (7b) at the free end leads to the

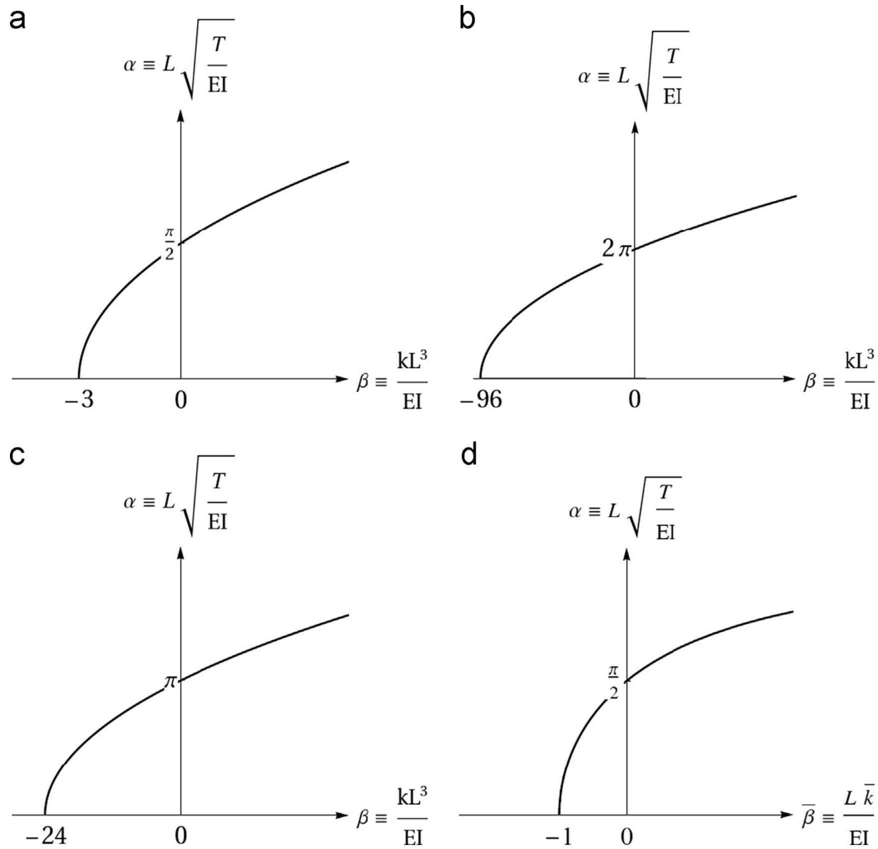

Fig. 3. Relation between the critical buckling load and spring resilience. (a) Cantilever (linear spring), (b) clamped–clamped (linear spring), (c) pinned–pinned (linear spring), (d) cantilever (rotary spring).

system of equations:

$$\begin{bmatrix} \cos(pL) & -\sin(pL) \\ q[\cos(pL)-1] & -p^3+q[pL-\sin(pL)] \end{bmatrix} \begin{bmatrix} A \\ D \end{bmatrix} = 0. \tag{8}$$

A non-trivial solution (5) requires that (A, D) do not vanish simultaneously, and implies that the determinant of the matrix vanishes, leading to

$$p^3 \cos(pL) = q[pL \cos(pL) - \sin(pL)]. \tag{9}$$

This is the relation $T(k)$ between the critical buckling load (2a) and the resilience of the spring (7a).

In the absence of a spring (10a), the smallest root of (10b) is (10c):

$$q=0: \quad \cos(pL)=0, \quad pL=\frac{\pi}{2}, \quad T_1(0)=\frac{\pi^2 EI}{4L^2}, \tag{10a–d}$$

and specifies the well-known critical buckling load (10d). The critical buckling load (10d) is modified by the presence of the spring because (10b) is not a root of (9) if $q \neq 0$. In the presence of the spring, the critical buckling load is given by (9) $\equiv$ (11):

$$p^3 = q[pL - \tan(pL)]. \tag{11}$$

The buckling relation (11) relating the axial tension (2a) to the resilience of the spring (7a) will be discussed subsequently (Section 4) for all values of parameters (p, q); a few examples for particular values of (p, q) are given next. An example far removed from the case (10a–d) is a critical buckling load four times larger (12a) implying (12b):

$$T_1(k_a)=\pi^2\frac{EI}{L^2}; \quad pL=\pi; \quad q=\frac{p^2}{L}=\frac{\pi^2}{L^3}, \quad k_a=\frac{\pi^2 EI}{L^3}, \tag{12a–d}$$

this implies (11) by (12c) a spring resilience (7a) that is positive (12d) because the spring opposes bending. Another example far removed from (10a–d) is the case of critical buckling load four times smaller (13a) implying (13b):

$$T_1(k_b)=\frac{\pi^2 EI}{16L^2}; \quad pL=\frac{\pi}{4}; \tag{13a, b}$$

$$q=\frac{p^3}{pL-1}=\frac{1}{16L^3}\frac{\pi^3}{\pi-4}, \quad k_b=-\frac{\pi^3 EI}{16(4-\pi)L^3}, \tag{13c, d}$$

this corresponds (11) to (13c) leading (7a) to a negative spring resilience (13d) showing that the spring favours the bending.

The relation between the buckling load and spring resilience (11) can be put into a dimensionless form:

$$\alpha \equiv pL = L\sqrt{\frac{T}{EI}}, \quad \beta \equiv qL^3 = \frac{kL^3}{EI}: \quad \alpha^3 = \beta(\alpha - \tan\alpha), \tag{14a–c}$$

that is plotted in Fig. 3a. Using the leading terms of the power series [6] for the tangent (15a) implies (15b):

$$\tan\alpha = \alpha + \frac{\alpha^3}{3} + O(\alpha^5): \quad \lim_{\alpha\to 0}\beta = \lim_{\alpha\to 0}\frac{\alpha^3}{\alpha-\tan\alpha} = -3. \tag{15a, b}$$

This proves that the beam will buckle without an axial load (16a) corresponding to (15b) $\equiv$ (16a,b):

$$T_1(k_c)=0: \quad \alpha=0, \quad \beta=-3, \quad k_c=-\frac{3EI}{L^3}, \tag{16a–d}$$

if the critical spring resilience (7a) is (16d). The plot of (14c) in Fig. 3a includes the preceding 3 cases (10a–d), (13a–d) and (16a–d) as well as intermediate values, and concerns a cantilever beam; it is a part of the complete plot for all values of (p, q) discussed subsequently (Section 4.4). The shape of the elastica of the buckled clamped–free beam with linear spring at the free end is given, to within a multiplying constant, by (17b):

$$D = A\cot(pL): \quad \zeta(x) = A\left[1 - \cos(px) + \frac{\sin(px) - (px)}{\tan(pL)}\right], \tag{17a, b}$$

that is obtained substituting (17a) in (5); the first line of (8) coincides with (17a). The same method applies to other cases of support, for example clamped at both ends that is considered next.

2.2. Clamped–clamped beam with spring at the middle

Since the beam is clamped at the end $x=0$, the equation of the elastica (5) remains valid. The shape of the elastica is symmetric and thus the slope vanishes at the middle (18a) where the spring is applied (18b):

$$\zeta'\left(\frac{L}{2}\right) = 0, \quad \zeta'''\left(\frac{L}{2}\right) + p^2\zeta'\left(\frac{L}{2}\right) = q\zeta\left(\frac{L}{2}\right). \tag{18a, b}$$

Substituting the shape of the elastica (5) in the boundary conditions (18a,b) leads to the system of equations:

$$\begin{bmatrix} \sin\left(\frac{pL}{2}\right) & \cos\left(\frac{pL}{2}\right) - 1 \\ q\left[\cos\left(\frac{pL}{2}\right) - 1\right] & -p^3 + q\left[\frac{pL}{2} - \sin\left(\frac{pL}{2}\right)\right] \end{bmatrix}\begin{bmatrix} A \\ D \end{bmatrix} = 0. \tag{19}$$

The relation between the critical buckling load and spring resilience is specified by the vanishing of the determinant of the matrix in (19), leading to

$$p^3\sin\left(\frac{pL}{2}\right) = q\left[\frac{pL}{2}\sin\left(\frac{pL}{2}\right) + 2\cos\left(\frac{pL}{2}\right) - 2\right]. \tag{20}$$

In the absence of spring (21a), the smallest root of (21b) is (21c), leading to the well known critical load buckling load (21d) for a clamped–clamped beam:

$$q = 0: \quad \sin\left(\frac{pL}{2}\right) = 0, \quad pL = 2\pi, \quad T_2(0) = 4\pi^2\frac{EI}{L^2} = 16T_1(0), \tag{21a–d}$$

that is 16 times that of the cantilever beam (10d).

An example far removed from (21a–d) is a critical buckling load four times smaller (22a) implying (22b):

$$T_2(k_d) = \pi^2\frac{EI}{L^2}; \quad pL = \pi; \tag{22a, b}$$

$$q = \frac{2p^3}{\pi - 4} = \frac{2\pi^3}{(\pi - 4)L^3}, \quad k_d = -\frac{2\pi^3 EI}{(4 - \pi)L^3}, \tag{22c, d}$$

this corresponds (20) to (22c) leading (7a) to a negative spring resilience (21d) implying that the spring favours bending. Using (14a,b), the relation (20) between the critical buckling load (14a) and the resilience of the spring (14b) can be put in the dimensionless form:

$$\alpha^3\sin\left(\frac{\alpha}{2}\right) = \beta\left[\frac{\alpha}{2}\sin\left(\frac{\alpha}{2}\right) + 2\cos\left(\frac{\alpha}{2}\right) - 2\right]. \tag{23}$$

Using [7] the power series:

$$\sin\left(\frac{\alpha}{2}\right) = \frac{\alpha}{2} - \frac{1}{3!}\left(\frac{\alpha}{2}\right)^3 + O(\alpha^5), \tag{24a}$$

$$\cos\left(\frac{\alpha}{2}\right) = 1 - \frac{1}{2}\left(\frac{\alpha}{2}\right)^2 + \frac{1}{4!}\left(\frac{\alpha}{2}\right)^4 + O(\alpha^6), \tag{24b}$$

the term in square brackets in (23) scales for small α as

$$\frac{\alpha^3}{\beta}\sin\left(\frac{\alpha}{2}\right) = \left(\frac{\alpha}{2}\right)^2\left(1 - \frac{\alpha^2}{24}\right) + 2\left(1 - \frac{\alpha^2}{8} + \frac{\alpha^4}{384}\right) - 2 + O(\alpha^6) = -\frac{\alpha^4}{192} + O(\alpha^6), \tag{25}$$

implying the limit:

$$\lim_{\alpha\to 0}\beta = -\lim_{\alpha\to 0}\frac{192}{\alpha}\sin\left(\frac{\alpha}{2}\right) = -96. \tag{26}$$

Thus buckling occurs in the absence of an axial load (27a) for a critical spring resilience (27b):

$$T_2(k_e) = 0: \quad k_e = -\frac{96EI}{L^3}. \tag{27a, b}$$

The relation (23) between the critical buckling load (14a) and the resilience of the spring (14b) is plotted in Fig. 3b. The shape of the elastica of the buckled clamped–clamped beam with a linear spring at the middle is given, to within a multiplying constant, by (28b):

$$A = \left[\cot\left(\frac{pL}{2}\right) - \csc\left(\frac{pL}{2}\right)\right]D: \tag{28a}$$

$$\zeta(x) = D\left\{\sin(px) - (px) + \left[\cot\left(\frac{pL}{2}\right) - \csc\left(\frac{pL}{2}\right)\right][1 - \cos(px)]\right\}, \tag{28b}$$

that is obtained substituting (28a) in (5); the first line of (19) coincides with (28a). The replacement of clamped by pinned supports leads to smaller critical buckling loads for the same spring resilience that are considered next.

2.3. Replacement of clamped by pinned supports

The boundary conditions at the pinned support at $x=0$ relate the constants of integration in (3):

$$0 = \zeta(0) = A + C, \quad 0 = \zeta''(0) = -p^2 C, \tag{29a, b}$$

leading to the shape of the elastica:

$$\zeta(x) = Bx + D\sin(px). \tag{30}$$

The shape of the elastica is symmetric and thus the same boundary conditions (18a,b) with spring in the middle can be applied:

$$\begin{bmatrix} 1 & p\cos\left(\frac{pL}{2}\right) \\ \frac{qL}{2} - p^2 & q\sin\left(\frac{pL}{2}\right) \end{bmatrix}\begin{bmatrix} B \\ D \end{bmatrix} = 0. \tag{31}$$

The vanishing of the determinant specifies the relation between the buckling load and the resilience of the spring:

$$p^3\cos\left(\frac{pL}{2}\right) = q\left[\frac{pL}{2}\cos\left(\frac{pL}{2}\right) - \sin\left(\frac{pL}{2}\right)\right]. \tag{32}$$

The buckling relation for a pinned–pinned beam (32) is similar to that for a cantilever beam (9) replacing L by $L/2$; both are analysed in detail subsequently (Section 4). In the absence of spring (33a), the smallest positive root of (33b) specifies (33c) the well known critical buckling load for a pinned–pinned beam:

$$q = 0: \quad \cos\left(\frac{pL}{2}\right) = 0, \quad pL = \pi, \quad T_3(0) = \pi^2\frac{EI}{L^2} = 4T_1(0) = \frac{1}{4}T_2(0), \tag{33a–d}$$

that is four times larger (smaller) than in the clamped–free (10d) [clamped–clamped (21d)] case.

An example far removed from (33a–d) is a critical buckling load four times larger (34a) implying (34b):

$$T_3(k_f) = 4\pi^2\frac{EI}{L^2}; \quad pL = 2\pi; \quad q = \frac{p^3}{\pi} = \frac{8\pi^2}{L^3}, \quad k_f = \frac{8\pi^2 EI}{L^3}, \tag{34a–d}$$

this corresponds (32) to (34c) to a spring resilience (34d) that is positive implying that the spring opposes bending. The relation (32) between the critical buckling load (14a) and the resilience of the spring (14b) can be put into the dimensionless form:

$$\alpha^3\cos\left(\frac{\alpha}{2}\right) = \beta\left[\frac{\alpha}{2}\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right]. \tag{35}$$

In the limit of small α, the term in square brackets scales (24b) as

$$\frac{\alpha^3}{\beta}\cos\left(\frac{\alpha}{2}\right)=\frac{\alpha}{2}\left(1-\frac{\alpha^2}{8}\right)-\frac{\alpha}{2}\left(1-\frac{\alpha^2}{24}\right)+O(\alpha^5)=-\frac{\alpha^3}{24}+O(\alpha^5). \quad (36)$$

This leads to the limit:

$$\lim_{\alpha\to 0}\beta=-\lim_{\alpha\to 0}24\cos\left(\frac{\alpha}{2}\right)=-24, \quad (37)$$

implying that buckling occurs without an axial load (38a) for (38b) spring resilience (38c):

$$T_3(k_g)=0:\quad \beta=-24,\quad k_g=-\frac{24EI}{L^3}, \quad (38a\text{–}c)$$

that favours bending, and in modulus is smaller (larger) than for the clamped–clamped (27b) [clamped–free (16d)] case. The relation (35) between the critical buckling load (14a) and the resilience of the spring (14b) is plotted in Fig. 3c. The shape of the elastica of the buckled pinned–pinned beam with linear spring at the middle is given, to within a multiplying constant, by (39b):

$$B=pD\cos\left(\frac{pL}{2}\right):\quad \zeta(x)=D\left[\sin(px)+px\cos\left(\frac{pL}{2}\right)\right], \quad (39a, b)$$

that is obtained substituting (39a) in (28b); the first line of (31) coincides with (39a). The remaining combination of supports is the clamped–pinned beam considered next.

2.4. Clamped–pinned beam with spring at the middle

Since the shape is not symmetric, the elastica (3) is given: (i) in the lower half by (5) ≡ (40a) that satisfies the boundary conditions (4a,b) at the clamped end:

$$\zeta\left(x\le\frac{L}{2}\right)=A[1-\cos(px)]+D[\sin(px)-px]; \quad (40a)$$

$$\zeta\left(x\ge\frac{L}{2}\right)=B(L-x)+C\sin[p(L-x)], \quad (40b)$$

(ii) in the upper half by (30) ≡ (40b) that meets the boundary conditions (29a,b) at the pinned end replacing x by $L-x$. The four arbitrary constants are determined by four conditions at the matching point in the middle: (iii) continuity of the displacement and slope (41a,b);

$$\zeta\left(\frac{L}{2}-0\right)=\zeta\left(\frac{L}{2}+0\right),\quad \zeta'\left(\frac{L}{2}-0\right)=\zeta'\left(\frac{L}{2}+0\right); \quad (41a, b)$$

$$\zeta''\left(\frac{L}{2}-0\right)=\zeta''\left(\frac{L}{2}+0\right),\quad \zeta'''\left(\frac{L}{2}+0\right)-\zeta'''\left(\frac{L}{2}-0\right)=q\zeta\left(\frac{L}{2}\right); \quad (41c, d)$$

(iii) since there is no applied torque, the curvature is also continuous (41c); (iv) the transverse force has a jump (41d) due to the force of the spring. In the l.h.s of (41d), the term of the force can be omitted due to the continuity of the slope (41b) and, on the r.h.s. of (41d), the displacement is unique by (41a). Substituting the shape of the elastica (40a,b) in the boundary conditions (41a–d) leads to the system of equations:

$$\begin{bmatrix} 1-\cos\left(\frac{pL}{2}\right) & -\frac{L}{2} & -\sin\left(\frac{pL}{2}\right) & \sin\left(\frac{pL}{2}\right)-\frac{pL}{2} \\ p\sin\left(\frac{pL}{2}\right) & 1 & p\cos\left(\frac{pL}{2}\right) & p\cos\left(\frac{pL}{2}\right)-p \\ \cos\left(\frac{pL}{2}\right) & 0 & \sin\left(\frac{pL}{2}\right) & -\sin\left(\frac{pL}{2}\right) \\ -p^3\sin\left(\frac{pL}{2}\right) & -q\frac{L}{2} & -q\sin\left(\frac{pL}{2}\right)-p^3\cos\left(\frac{pL}{2}\right) & -p^3\cos\left(\frac{pL}{2}\right) \end{bmatrix}\begin{bmatrix} A\\ B\\ C\\ D\end{bmatrix}=0. \quad (42)$$

The vanishing of the determinant of the matrix in (42) specifies the relation between the critical buckling load and the resilience of the spring for the clamped–pinned beam. The shape of the elastica of beam with linear spring at the middle is given, to within a constant multiplying factor, by (40a,b) where the four constants of integration can be expressed in terms of one using (42).

3. Replacement of the linear by a rotary spring

A linear (rotary) spring affects the critical buckling load by applying a force (moment) proportional to the displacement (slope); the same four combinations of support [Section 2 (Section 3)] can be considered, namely clamped-free [Section 2.1 (Section 3.1)], clamped–clamped [Section 2.2 (Section 3.2)], pinned–pinned [Section 2.3 (Section 3.3)] and clamped–pinned [Section 2.4 (Section 3.4)].

3.1. Rotary spring at the free end of a cantilever beam

If the linear spring is replaced by a rotary spring the boundary conditions (6a,b) are replaced by: (i) a bending moments (43a) proportional to the slope through the resilience of the spring; (ii) zero transverse force (43b):

$$EI\zeta''(L)=-\overline{k}\zeta'(L),\quad EI\zeta'''(L)+T\zeta'(L)=0. \quad (43a, b)$$

The boundary conditions (43a,b) ≡ (44b,c) involve two parameters:

$$\overline{q}=\frac{\overline{k}}{EI}:\quad \zeta''(L)+\overline{q}\zeta'(L)=0,\quad \zeta'''(L)+p^2\zeta'(L)=0, \quad (44a\text{–}c)$$

namely (2a) and the ratio (44a) of the resilience of the rotary spring to the bending stiffness. The shape of the elastica (3) for a beam clamped (4a,b) at $x=0$ is (5). Substituting the shape (5) of the elastica in the boundary conditions (44b,c) leads to the system of equations:

$$\begin{bmatrix} p^2\cos(pL)+p\overline{q}\sin(pL) & -p^2\sin(pL)+p\overline{q}\cos(pL)-p\overline{q} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A\\ D\end{bmatrix}=0. \quad (45)$$

The relation between the critical buckling load and resilience of the rotary spring is

$$p\cos(pL)=-\overline{q}\sin(pL). \quad (46)$$

In the absence of spring $\overline{q}=0$, this leads to the same critical buckling load (10a–d) as before.

An example far removed from (10a–d) is a critical buckling load four times larger (47a) implying (47b):

$$T_1(\overline{k}_a)=\pi^2\frac{EI}{L^2},\quad pL=\pi;\qquad \overline{q}=\infty,\quad \overline{k}_a=\infty, \quad (47a\text{–}d)$$

this would require (46) an infinite (47c) resilience (47d) of the rotary spring. An example far removed from (10a–d) in the opposite direction is a critical buckling load four times smaller (48a) implying (48b):

$$T_1(\overline{k}_b)=\frac{\pi^2}{16}\frac{EI}{L^2},\quad pL=\frac{\pi}{4};\qquad \overline{q}=-p=-\frac{\pi}{4L},\quad \overline{k}_b=-\frac{\pi EI}{4L}, \quad (48a\text{–}d)$$

this corresponds (46) to (48c) the resilience (48d) of the rotary spring. The relation (46) between the critical buckling load (14a) and the resilience of the rotary spring (49a) is (49b):

$$\overline{\beta}=\frac{\overline{k}L}{EI}:\quad \overline{\beta}=-\alpha\cot\alpha. \quad (49a, b)$$

The buckling in the absence of an axial load (50a) occurs (50b) for the resilience (50c) of the rotary spring:

$$\lim_{\alpha\to 0}\overline{\beta}=\lim_{\alpha\to 0}-\frac{\alpha}{\tan\alpha}=-1,\quad T_1(\overline{k}_c)=0,\quad \overline{k}_c=-\frac{EI}{L}. \quad (50a\text{–}c)$$

The relation (49b) between the critical buckling load (14a) and the resilience of the spring (49a) is plotted in Fig. 3d. The shape of the elastica of the buckled clamped-free beam with a rotary spring at the free end is given, to within a multiplying constant, by (51b):

$$D=0: \quad \zeta(x)=A[1-\cos(px)], \tag{51a, b}$$

that is obtained substituting (51a) in (5); the second line of (45) coincides with (51a).

3.2. Rotary spring at the middle of a clamped–clamped beam

The rotary spring can cause a skew-symmetry that breaks the symmetry of the shape of the elastica of the buckled beam. Thus the shape of the elastica is given: (i) in the lower half by (5) ≡ (52a) that meets the boundary condition (4a,b) of clamping at $x=0$:

$$\zeta\left(x\le\frac{L}{2}\right)=A[1-\cos(px)]+D[\sin(px)-px]; \tag{52a}$$

$$\zeta\left(x\ge\frac{L}{2}\right)=B\{1-\cos[p(L-x)]\}+C\{\sin[p(L-x)]-p(L-x)\}, \tag{52b}$$

(ii) in the upper half by (52b) that replaces x by $L-x$ to satisfy the clamping condition at $x=L$. At the matching point in the middle $x=L/2$; (i,ii) the displacement and slope are continuous (41a,b); (iii) since there is no transverse force, the third derivative is also continuous (53a); (iv) the bending moment has a jump (53b) due to the rotary spring:

$$\zeta'''\left(\frac{L}{2}-0\right)=\zeta'''\left(\frac{L}{2}+0\right), \quad \zeta''\left(\frac{L}{2}-0\right)-\zeta''\left(\frac{L}{2}+0\right)=\overline{q}\zeta'\left(\frac{L}{2}\right). \tag{53a, b}$$

Substituting (52a,b) in the matching conditions (41a,b); (53a,b) leads to

The relation between the critical buckling load (14a) and the resilience of the rotary spring (49a) is specified for the clamped–clamped beam by the vanishing of the determinant of the matrix (54). The shape of the elastica of a buckled clamped–clamped beam with a rotary spring in the middle is given, to within a multiplying constant, by (52a,b) where all four constants of integration can be expressed in terms of one of them using (54).

3.3. Replacement of clamped by pinned supports

The symmetry under buckling is again violated by the skew-symmetry of the rotary spring, and the shape of the elastica is given: (i) in the lower half by (30) ≡ (55a) that meets the pinning boundary conditions at $x=0$:

$$\begin{bmatrix} 1-\cos\left(\frac{pL}{2}\right) & \cos\left(\frac{pL}{2}\right)-1 & \frac{pL}{2}-\sin\left(\frac{pL}{2}\right) & \sin\left(\frac{pL}{2}\right)-\frac{pL}{2} \\ p\sin\left(\frac{pL}{2}\right) & -p\sin\left(\frac{pL}{2}\right) & -p-p\cos\left(\frac{pL}{2}\right) & -p+p\cos\left(\frac{pL}{2}\right) \\ \sin\left(\frac{pL}{2}\right) & \sin\left(\frac{pL}{2}\right) & \cos\left(\frac{pL}{2}\right) & \cos\left(\frac{pL}{2}\right) \\ p^2\cos\left(\frac{pL}{2}\right)-p\overline{q}\sin\left(\frac{pL}{2}\right) & -p^2\cos\left(\frac{pL}{2}\right) & p^2\sin\left(\frac{pL}{2}\right) & -p^2\sin\left(\frac{pL}{2}\right)+p\overline{q}-p\overline{q}\cos\left(\frac{pL}{2}\right) \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}=0. \tag{54}$$

$$\zeta\left(x\le\frac{L}{2}\right)=Bx+D\sin(px); \tag{55a}$$

$$\zeta\left(x\ge\frac{L}{2}\right)=A(L-x)+C\sin[p(L-x)], \tag{55b}$$

(ii) in the upper half (55b) substituting x by $L-x$. Substituting the shape of the elastica (55a,b) in the matching conditions (41a,b); (53a,b) leads to the system of equations:

$$\begin{bmatrix} -\frac{L}{2} & \frac{L}{2} & -\sin\left(\frac{pL}{2}\right) & \sin\left(\frac{pL}{2}\right) \\ 1 & 1 & p\cos\left(\frac{pL}{2}\right) & p\cos\left(\frac{pL}{2}\right) \\ 0 & 0 & \cos\left(\frac{pL}{2}\right) & \cos\left(\frac{pL}{2}\right) \\ 0 & \overline{q} & -p^2\sin\left(\frac{pL}{2}\right) & p^2\sin\left(\frac{pL}{2}\right)+p\overline{q}\cos\left(\frac{pL}{2}\right) \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}=0. \tag{56}$$

The relation between the critical buckling load (14a) and the resilience of the rotary spring (49a) is specified for the pinned–pinned beam by the vanishing of the determinant of the matrix in (56). The shape of the elastica of a buckled pinned–pinned beam with a rotary spring in the middle is given, to within a multiplying constant, by (55a,b) where the four constants of integration can be expressed in terms of one of them using (56).

3.4. Clamped–pinned beam with rotary spring

The shape of the elastica is not symmetric because: (i) the clamping and pinning boundary conditions are distinct; (ii) the rotary spring in the middle violates symmetry by adding skew-symmetry. The shape of the elastica is given by (40a) in the lower and (40b) in the upper half. The matching conditions at the location $x=L/2$ of the rotary spring are (41a,b); (53a,b). Substituting the former (40a,b) in the latter (41a,b); (53a,b) leads to the system of equations:

$$\begin{bmatrix} 1-\cos\left(\frac{pL}{2}\right) & -\frac{L}{2} & -\sin\left(\frac{pL}{2}\right) & \sin\left(\frac{pL}{2}\right)-\frac{pL}{2} \\ p\sin\left(\frac{pL}{2}\right) & 1 & p\cos\left(\frac{pL}{2}\right) & p\cos\left(\frac{pL}{2}\right)-p \\ \sin\left(\frac{pL}{2}\right) & 0 & \cos\left(\frac{pL}{2}\right) & \cos\left(\frac{pL}{2}\right) \\ p^2\cos\left(\frac{pL}{2}\right) & \overline{q} & p^2\sin\left(\frac{pL}{2}\right)+p\overline{q}\cos\left(\frac{pL}{2}\right) & -p^2\sin\left(\frac{pL}{2}\right) \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}=0. \tag{57}$$

The relation between the critical buckling load (14a) and the resilience of the rotary spring (49a) for a clamped–pinned beam is specified by the roots of the determinant of the matrix (57). The shape of the elastica of a buckling clamped–pinned beam with a rotary spring at the middle is given, to within a multiplying constant, by (40a,b) where the four constants of integration can be expressed in terms of one of them using (57).

4. Relation between buckling load and spring resilience

The buckling relation between the critical axial load (14a) and the resilience of a linear (14b) or rotary (49a) spring has been obtained in all 8 cases of: (i–iv) clamped–pinned beam with a linear spring in the middle (42), and clamped–clamped (54), pinned–pinned (56) and clamped–pinned (57) beam with a rotary spring in the middle;

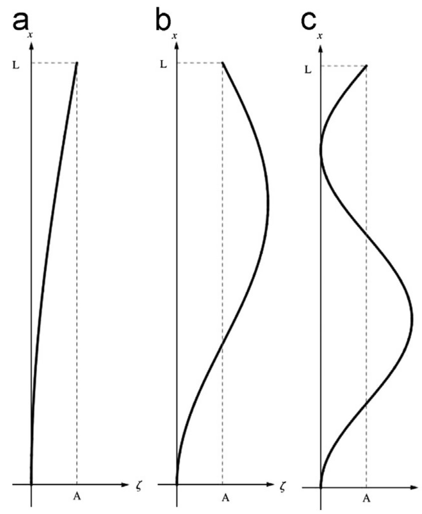

Fig. 4. First three modes of buckled cantilever beam. (a) 1st mode, (b) 2nd mode and (c) 3rd mode.

(v–viii) the cantilever beam with linear (14c) or rotary (49b) spring at the tip, and clamped–clamped (23) and pinned–pinned (35) beam with a linear spring in the middle. In the cases (v–viii), the relation between axial tension and spring resilience was illustrated respectively in Figs. 3a–d near the critical spring resilience that leads to buckling without axial load. The relation between axial load and spring resilience is considered (Section 4) for all buckling modes (Section 4.1) for the case of a cantilever beam with a linear (Section 4.2) or rotary (Section 4.3) spring at the tip.

4.1. Buckling modes for a cantilever beam

The buckling relation specifying the critical axial buckling load (14a) for a cantilever beam is (49b) ≡ (58):

$$\frac{\tan\,\alpha}{\alpha} = -\frac{1}{\overline{\beta}}, \tag{58}$$

for a rotary spring (49a), and (14c) ≡ (59):

$$\frac{\tan\,\alpha}{\alpha} = 1 - \frac{\alpha^2}{\beta} \equiv s, \tag{59}$$

for a linear spring (14b), both cases with the spring at the tip. In the case of a pinned–pinned beam with a linear spring in the middle, the buckling relation (35) ≡ (60),

$$\frac{\tan\,(\alpha/2)}{\alpha/2} = 1 - \frac{8}{\beta}\left(\frac{\alpha}{2}\right)^2, \tag{60}$$

is similar to (59) with the substitution $\alpha \to \alpha/2$ and $\beta \to \beta/8$. Since the case (60) is reducible to (59), only the latter will be considered in the sequel. In the absence of either linear (61a) or rotary (61b) spring, respectively, (59) and (58) both imply (61c) that there is an infinity of buckling modes (61d),

$$\beta = 0 \text{ or } \overline{\beta} = 0: \quad \tan\,\alpha = \infty \Rightarrow \alpha_n = n\pi - \frac{\pi}{2}, \tag{61a–d}$$

with (62a). From respectively (17b) and (51) follows the shape (62c) of the modes (62b):

$$n = 1, 2, 3, \ldots; \; p_n = \frac{\alpha_n}{L}: \quad \zeta_n(x) = A\left\{1 - \cos\left[\left(n - \frac{1}{2}\right)\frac{\pi x}{L}\right]\right\}. \tag{62a–c}$$

The first three $n = 1, 2, 3$ are illustrated in Fig. 4. All modes have amplitude A at the tip (63a),

$$\zeta_n(L) = A, \quad \zeta_n'(L) = A\left(n - \frac{1}{2}\right)\frac{\pi}{L}(-1)^n, \tag{63a, b}$$

and slope (63b) at the tip. The shape of the modes (62c) and the values (63a,b) are changed in the presence of the linear (rotary) spring since (62b) are longer roots of (59) [(58)]. The roots are considered next, first analytically (Section 4.2) and then graphically (Sections 4.3 and 4.4).

4.2. Infinite roots for the critical buckling load

The buckling relations for the cantilever beam with rotary (58) or linear (59) spring at the tip both involve the circular tangent whose MacLaurin series [6] is (64b):

$$|\alpha| < \frac{\pi}{2}: \qquad \tan\,\alpha = \frac{1}{\alpha}\sum_{n=1}^{\infty}(-1)^n \frac{1 - 2^{2n}}{(2n)!} B_{2n}(2\alpha)^{2n}, \tag{64a, b}$$

valid for (64a) and involving the Bernoulli numbers B_{2n} of which the first five [8] are

$$B_0 = 1, \quad B_2 = \tfrac{1}{6}, \quad B_4 = -\tfrac{1}{30} = B_8, \quad B_6 = \tfrac{1}{42}. \tag{65a–e}$$

Substituting (64b) in (58) gives the series

$$\begin{aligned} -\frac{1}{\overline{\beta}} &= \sum_{n=1}^{\infty}(-1)^n \frac{1 - 2^{2n}}{(2n)!} B_{2n} 2^{2n} \alpha^{2n-2} \\ &= 1 + \frac{\alpha^2}{3} + \frac{2}{15}\alpha^4 + \frac{34}{105}\alpha^6 + o(\alpha^8) = G \prod_{n=1}^{\infty}(\alpha - \alpha_n), \end{aligned} \tag{66}$$

that agrees with (15a) and (50a) at lowest order; $G \neq 0$ in (66) whose roots specify the critical axial loads for all the buckling modes of a cantilever beam with a rotary spring at the tip; in the case of a linear spring (64b) is substituted in (59) leading to

$$\begin{aligned} -\frac{1}{\beta} &= \frac{\tan\,\alpha - \alpha}{\alpha^3} = \frac{1}{\alpha^4}\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1 - 2^{2n}}{(2n)!} B_{2n}(2\alpha)^{2n} \\ &= -\frac{1}{3} + \frac{2}{15}\alpha^2 + \frac{17}{315}\alpha^4 + o(\alpha^6) = H \prod_{n=1}^{\infty}(\alpha - \alpha_n), \end{aligned} \tag{67}$$

that agrees with (15b) to lowest order, and where $H \neq 0$. The roots of (66) and (67) can be calculated approximately by truncating the series. They can be visualised graphically by a method applied next, first to the case of a rotary (Section 4.3) and then to the case of a linear (Section 4.4) spring opposing or favouring the buckling of the cantilever beam.

4.3. Rotary spring favouring or opposing buckling

The buckling relation for the cantilever beam with a rotary spring (58) involves the circular tangent that is plotted in Fig. 5

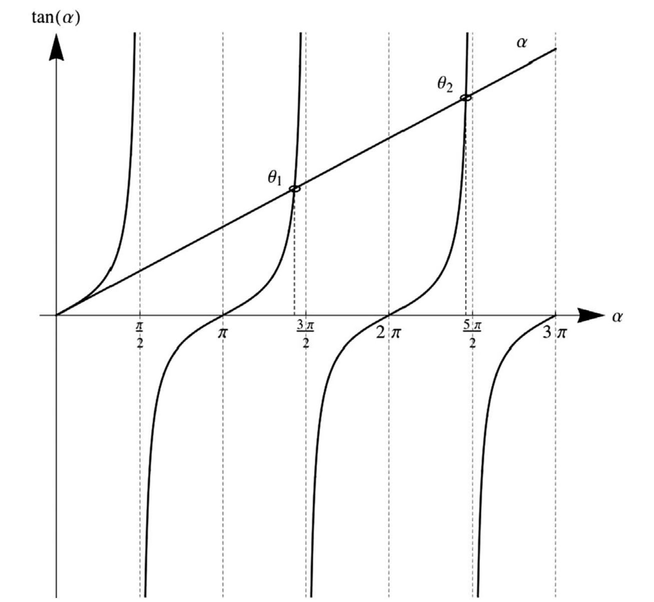

Fig. 5. Plots of the circular tangent and its variable to determine their ratio and the first point $\theta_1 = 4.493$ where they are equal.

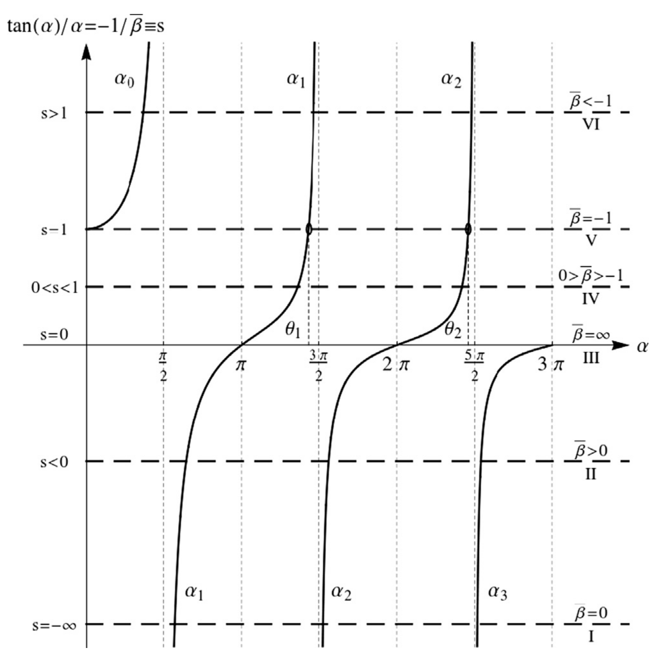

Fig. 6. Roots α_n of the critical buckling load (14a) versus the resilience of a rotary spring (49a) at the tip of a cantilever beam (58).

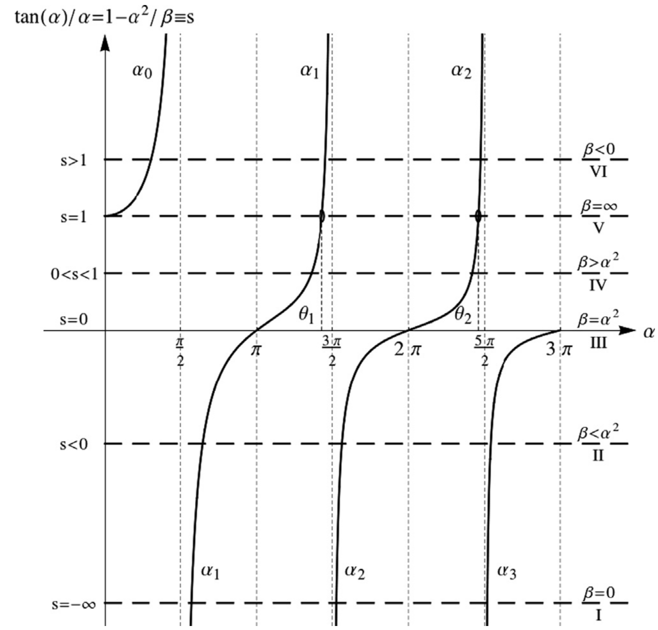

Fig. 7. Roots α_n of the critical buckling load (14a) versus the resilience of a linear spring (14b) at the tip of a cantilever beam (59).

showing where it equals its variable, represented by the diagonal of the quadrant. The smallest roots of (68a) are recorded (68b–d) for future use:

$$\tan\,\theta_n = \theta_n: \quad \theta_1 = 4.493, \quad \theta_2 = 7.725, \quad \theta_3 = 10.904. \tag{68a–d}$$

The ratio of the circular tangent to its variable, represented by the function $s \equiv \alpha^{-1}\,\tan\,\alpha$, is plotted in Fig. 6 and the roots for the critical buckling load are the intersections with the horizontal lines of constant spring resilience, leading (Table 2) to six cases: (I) in the absence of rotary spring, $\overline{\beta} = 0$, the roots for $s = -\infty$ are (61d);(II) if the rotary opposes bending, $\overline{\beta} > 0$, the roots lie in the range $n\pi - \pi/2 < \alpha_n < n\pi$ implying an increase in the critical buckling load;(III) for an infinitely strong rotary spring opposing bending, $\overline{\beta} = \infty$, the critical buckling loads are $\alpha_n = n\pi$;(IV) if the rotary spring weakly favours bending, $0 > \overline{\beta} > -1$, the critical buckling load still increases $n\pi < \alpha_n < \theta_n$ where θ_n is the n-th root of (68a);(V) for a spring with transition resilience, $\overline{\beta} = -1$, a new root α_0 appears, leading to buckling without an axial load, in agreement with (50a–c);(VI) a stronger rotary spring favouring bending, $\overline{\beta} < -1$, the critical buckling load increases, $0 < \alpha_0 < \pi/2$ and $\theta_n < \alpha_n < n\pi + \pi/2$. Fig. 3d corresponds to the top left of Fig. 6, where (i) the root α_0 is the lowest-order buckling mode for a rotary spring favouring bending; (ii) the root α_1 is the lowest-order buckling mode for a rotary spring opposing bending. For a given critical buckling load α, there is only one possible buckling mode: (i) with a rotary spring favouring bending if $n\pi < \alpha_n < n\pi + \pi/2$ with $n = 0, 1, 2, \ldots$; (ii) for a rotary spring opposing bending if $n\pi - \pi/2 < \alpha < n\pi$ with $n = 1, 2, 3, \ldots$; (iii) there is a jump from one mode to the next for $\alpha = n\pi$. For a given spring resilience $\overline{\beta}$, an infinity of buckling modes with increasing critical buckling load is possible: (i) for a rotary spring favouring bending, the modes are α_n with n=0, 1, 2, …; (ii) for a rotary spring opposing bending for α_n with n=1, 2, 3, … . The preceding conclusions are modified for a linear spring at the tip of a cantilever beam because (58) is replaced by (59) where: (i) the l.h.s is the same; (ii) the r.h.s depends not only on β but also on α. Thus Fig. 6 is replaced next by Figs. 7 and 8 for the linear spring. Both the rotary and linear springs have atransition resilience with different values.

4.4. Transition resilience for buckling without an axial load

The buckling relation (59) for a cantilever beam with a linear spring at the tip is analysed in Fig. 7 by comparing the ratio of the circular tangent to its variable lines with the curves $1 - \beta/\alpha^2 \equiv s$ with the roots α_n as intersections, leading (Table 3) to six cases: (I) in the absence of spring (61a) the roots are (61d), of which the first $\alpha_1 = \pi/2$ corresponds to the critical buckling load (10c); (II) for a weak spring opposing bending, $\beta < \alpha^2$, the roots lie in the interval $n\pi - \pi/2 < \alpha_n < n\pi$ corresponding to higher critical buckling loads; (III) for a critical spring resilience:

$$\beta = \alpha^2 \Leftrightarrow q = p^2/L \quad \Leftrightarrow \quad s = 0, \tag{69a–c}$$

the roots are $\alpha_n = n\pi$; (IV) for a strong spring still opposing bending, $\beta > \alpha^2$, the roots lie in the range $n\pi < \alpha_n < \theta_n$, where θ_n is a root of (68a), corresponding to a further increase in the critical

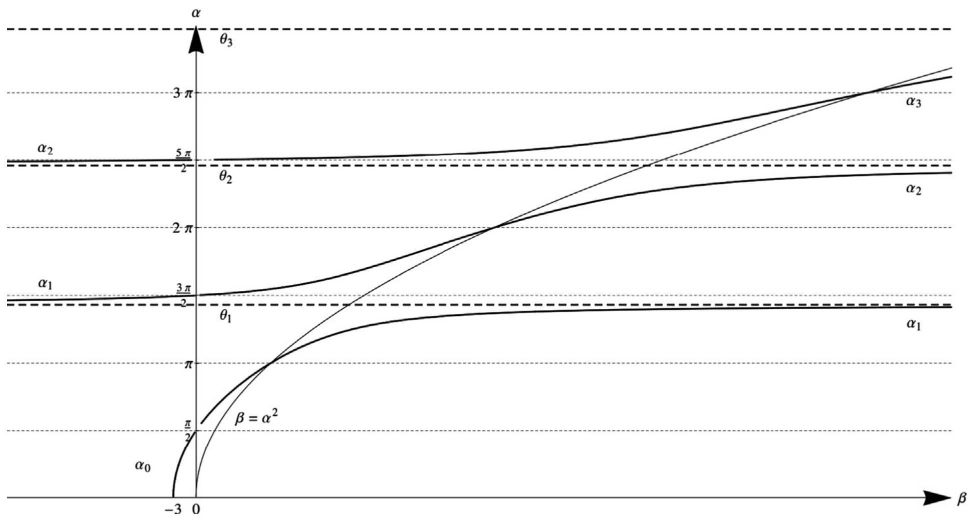

Fig. 8. Alternate form of Fig. 7 with α vertically and $s = 1 - \alpha^2/\beta$ replaced by β horizontally.

buckling load that can never exceed $p_n L < \theta_n$; (V) an infinitely strong spring, $\beta = \infty$, causes buckling with zero axial load because a new root α_0 appears, and the remaining roots are $\alpha_n = \theta_n$ for the higher-order buckling modes; (VI) a spring favouring bending, $\beta < 0$, leads to a lowest-order buckling mode with $0 < \alpha_0 < \pi/2$ and higher-order buckling modes with $\theta_n < \alpha_n < n\pi + \pi/2$. Fig. 6 (Fig. 7) and Table 2 (Table 3) for the cantilever beam with a rotary (linear) spring at the tip look similar, but there is an important difference: (i) in Fig. 6, the horizontal lines $\overline{\beta} =$ const do not depend on α; (ii) in Fig. 7, the horizontal lines $1 - \alpha^2/\beta =$ const depend both on β and α. In the case (ii), Fig. 7 is transformed to Fig. 8 separating α and βalong the two axes.

For (i) a linear spring opposing bending, all roots start at $\alpha_n = n\pi - \pi/2$ with $n = 1, 2, 3, \ldots$ and the critical buckling load increases with the resilience of the spring $\beta > 0$ passing through $\alpha_n = n\pi$ for $\beta = (\alpha_n)^2$ and reaching an asymptotic $\beta \to \infty$ maximum $\alpha_n = \theta_n$ where θ_n is the n-th root of (68a); (ii) for a linear spring favouring bending, $\beta < 0$, the roots α_n start at $\alpha_n = n\pi + \pi/2$ with $n = 1, 2, 3, \ldots$ and as $|\beta| = -\beta$ increases, the critical buckling load decreases to the asymptotic $\beta \to -\infty$ limit $\alpha_n = \theta_{n-1}$, where θ_{n-1} is the $(n-1)$-th root of (68a); (iii) thus there is a jump of 2π for the same root between a linear spring opposing and favouring bending. Another difference is that for a linear spring favouring bending, there is an additional lowest-order root α_0 that: (i) allows buckling without an axial load $\alpha_0 = 0$ for $\beta = -3$ in agreement with (16a–d); (ii) for $-3 < \beta < 0$, the critical buckling load increases in the range $0 < \alpha_0 < \pi/2$. Fig. 3a corresponds to the bottom left of Fig. 8 that includes all buckling modes. For a given critical buckling load α, there is only one mode: (i) for $n\pi - \pi/2 < \alpha < \theta_n$, it corresponds to a linear spring opposing bending; (ii) for $\theta_n < \alpha < n\pi + \pi/2$, it corresponds to a linear spring favouring bending, with $n = 1, 2, 3, \ldots$ in both cases; (iii) for a linear spring favouring bending, there is an additional lower-order mode $0 < \alpha_0 < \pi/2$. For a given resilience of the spring, there is an infinity of buckling modes α_n: (i) with $n = 1, 2, 3, \ldots$ for a linear spring opposing bending; (ii) with $n = 0, 1, 2, \ldots$ for a spring favouring bending.

5. Discussion

In all the preceding cases of buckling of a beam with any combination of supports and linear or rotary spring at the tip or middle, the roots α_n determine the buckled shape. For the cantilever beam with a rotary spring at the tip, the buckled shape

Table 1
Resilience of the spring that causes buckling without an axial load for the four simplest cases.

Case	Relation	Beam	Spring	Resilience
a	Fig. 3a	Cantilever	Linear	$k = -\frac{3EI}{L^3}$
b	Fig. 3b	Clamped–clamped	Linear	$k = -\frac{96EI}{L^3}$
c	Fig. 3c	Pinned–pinned	Linear	$k = -\frac{24EI}{L^3}$
d	Fig. 3d	Cantilever	Rotary	$\overline{k} = -\frac{EI}{L^3}$

is

$$\zeta_n(x) = A\left[1 - \cos\left(\alpha_n \frac{x}{L}\right)\right], \tag{70}$$

and the deflection at the tip (71a) may not be the maximum

$$\zeta_n(L) = A[1 - \cos(\alpha_n)], \quad \zeta_n'(L) = A\frac{\alpha_n}{L}\sin(\alpha_n), \tag{71a, b}$$

nor the slope at the tip (71b) has to be zero. Expressions for the shape and slope of the buckled cantilever beam with a linear spring at the tip can be obtained substituting (62b) in (17b).

The effect of a linear (rotary) spring on the buckling of a beam [Section 2 (Section 3)] has been considered for the four classical combinations of support: (i) clamped–free [Section 2.1 (Section 3.1)]; (ii) clamped–clamped [Section 2.2 (Section 3.2)]; (iii) pinned–pinned [Section 2.3 (Section 3.3)]; (iv) clamped–pinned [Section 2.4 (Section 3.4)]. The spring was placed at the tip in the case (i) and at the middle in the remaining cases (iii–iv). For each of the eight combinations was obtained: (i) the relation between the critical buckling load and the resilience of the spring; (ii) the resulting shape of the buckled elastica taking into account the effect of the linear or rotary spring.

Of the eight cases four are simplest (Table 1) leading to plots of the relation between the critical buckling load and the resilience of the spring: (a,d) the cantilever or clamped–free beam with linear (rotary) spring at the tip [Fig. 3a (d)]; (b,c) the clamped–clamped (pinned–pinned) beam with a linear spring at the middle [Fig. 3b (c)]. The linear or rotary spring can be placed so as to oppose (favour) bending [Fig. 1 (Fig. 2)] thus increasing (decreasing) the critical buckling load. Table 1 indicates, in the four simplest cases (a–d), the value of the resilience of the spring that causes buckling without an axial load.

Table 2
Critical buckling load for a cantilever beam with rotary spring.

Spring	None	Opposing bending		Favouring bending		
		Finite	Infinite	Weak	Transition	Strong
Resilience	$\overline{\beta}=0$	$\overline{\beta}>0$	$\overline{\beta}=\infty$	$0>\overline{\beta}>-1$	$\overline{\beta}=-1$	$\overline{\beta}<-1$
Case	I	II	III	IV	V	VI
Lowest root	$\alpha_1=\frac{\pi}{2}$	$\frac{\pi}{2}<\alpha_1<\pi$	$\alpha_1=\pi$	$\pi<\alpha_1<\theta_1$	$\alpha_0=0$	$0<\alpha_0<\frac{\pi}{2}$
Other roots	$\alpha_n=n\pi-\frac{\pi}{2}$	$n\pi-\frac{\pi}{2}<\alpha_n<n\pi$	$\alpha_n=n\pi$	$n\pi<\alpha_n<\theta_n$	$\alpha_n=\theta_n$	$\theta_n<\alpha_n<n\pi+\frac{\pi}{2}$

$\alpha\equiv pL=L\sqrt{T/EI}$; $\overline{\beta}=\overline{k}L/EI$; $n=1,2,\dots$.

Table 3
Critical buckling load for a cantilever beam with linear spring.

Spring	None	Opposing bending				Favouring bending
		Weak	Critical	Strong	Infinite	
Resilience	$\beta=0$	$\beta<\alpha^2$	$\beta=\alpha^2$	$\beta>\alpha^2$	$\beta=\infty$	$\beta<0$
Case	I	II	III	IV	V	VI
Lowest root	$\alpha_1=\frac{\pi}{2}$	$\frac{\pi}{2}<\alpha_1<\pi$	$\alpha_1=\pi$	$\pi<\alpha_1<\theta_1$	$\alpha_0=0$	$0<\alpha_0<\frac{\pi}{2}$
Other roots	$\alpha_n=n\pi-\frac{\pi}{2}$	$n\pi-\frac{\pi}{2}<\alpha_n<n\pi$	$\alpha_n=n\pi$	$n\pi<\alpha_n<\theta_n$	$\alpha_n=\theta_n$	$\theta_n<\alpha_n<n\pi+\frac{\pi}{2}$

$\alpha\equiv pL=L\sqrt{T/EI}$; $\beta=qL^3=kL^3/EI$; $n=1,2,\dots$.

Table 1 relates to Fig. 3 that shows the relation between the critical axial buckling load and the resilience of the spring near the critical case of buckling without axial load. For all possible combinations of axial buckling load and spring resilience, the buckling relation is less simple, as shown in detail for a cantilever beam (Section 4) with rotary (Section 4.3) or linear (Section 4.4) spring at the tip. The analysis in this case was made for all buckling modes without (Section 4.1) and with (Section 4.2) spring. The first three modes without spring are illustrated in Fig. 4, and the effect of the springs was demonstrated analytically (Section 4.2) and graphically (Sections 4.3 and 4.4). A common baseline graph (Fig. 5) illustrates the relation between critical axial load and spring resilience for a cantilever beam in the cases of rotary (Fig. 6 and Table 2) and linear (Figs. 7, 8 and Table 3) springs.

The present theory can be used in all eight cases in two ways: (i) the direct problem of determining the critical buckling load for a given resilience of the spring; (ii) the inverse problem of selecting the resilience of the spring so that the critical buckling load takes a desired value. The desired value of the critical buckling load may be: (α) an increase to resist buckling under higher axial load, by using a spring that opposes bending, thus increasing the load bearing capability of a structure or its safety margin; (β) a decrease to demolish a structure by buckling using a smaller axial load and a spring that favours bending, avoiding more extreme methods like controlled explosions or tedious disassembly.

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