ICAS 2016
30th Congress of the International Council of the Aeronautical Sciences
DCC, Daejeon, Korea ; September 25-30, 2016

ON AXISYMMETRIC ACOUSTIC-VORTICAL-ENTROPY WAVES IN PROPULSION SYSTEMS

L. M. B. C. Campos, A. C. Marta
Center for Aeronautical and Space Science and Technology, IDMEC, Instituto Superior Técnico,
Universidade de Lisboa, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal
luis.campos@tecnico.ulisboa.pt, andre.marta@tecnico.ulisboa.pt

Keywords: *aeroacoustics, absolute/convective instability, critical layers*

Abstract

The noise of jet and rocket engines involves the coupling of sound to swirling flows and to heat exchanges leading in the more complex cases of triple interactions to acoustic-vortical-entropy (AVE) waves. The present paper presents as far as the authors are aware the first derivation of the AVE equation for axisymmetric linear non-dissipative perturbations of a compressible, non-isentropic, swirling mean flow, with constant axial velocity and constant angular velocity. The axisymmetric AVE wave equation is obtained for the radial velocity perturbation, specifying its radial dependence for a given frequency and axial wavenumber. The AVE wave equation in the case of zero axial wavenumber has only one singularity at the sonic radius, where the isothermal Mach number for the swirl velocity is unity. The exact solution of the AVE wave equation is obtained as series expansions of Gaussian hypergeometric type valid inside, outside and around the sonic radius, thus: (i) covering the whole flow region; (ii) identifying the singularity at the sonic condition at the sonic radius; (iii) specifying near-axis and asymptotic solutions for small and large radius. Using polarization relations among wave variables specifies exactly the perturbations of: (i,ii) the radial and azimuthal velocity; (iii,iv) pressure and mass density; (v,vi) entropy and temperature. It is shown that the dependence of the AVE wave variables on the radial distance can be: (a) oscillatory with decaying amplitude; (b) monotonic with increasing amplitude. The case (b) of AVE wave amplitude increasing monotonically with the radial distance applies if the frequency times a function of the adiabatic exponent is less than the modulus of the vorticity (or twice the angular velocity). In the opposite case (a) the oscillatory nature of acoustic waves predominates over the tendency for monotonic growth of vortical perturbations. Associating sound with stable potential flows and swirl with unstable vortical flows suggests a criterion valid in non-isentropic conditions, that is in the presence of heat exchanges, that is a condition for stable combustion in a confined space: the peak vorticity (multiplied by a factor of order unity dependent on the adiabatic exponent) should be less than the lowest or fundamental frequency of the cavity.

1 Introduction

The noise of aircraft engines is a major limitation on airport operations, and the subject of ever more stringent certification rules, aiming to limit the total noise exposure as air traffic grows. The noise of the rocket engines of space launchers are sufficiently high to cause structural damage and require payloads like satellites to be tested in reverberant chambers. The literature on aircraft and rocket noise usually considers purely acoustic waves, although coupling with other modes occur in: (i) inlet ducts due to the shear flow in the wall boundary layers; (ii) in turbine exhausts due to the downstream swirling flow; (iii) in the combustion chambers and other heat generation and exchange processes involving non-isentropic flows. The simplest mean flow for which there are interacting acoustic, vortical and entropy perturbations is an axysymmetric non-isentropic flow with uniform axial velocity and rigid body swirl; this sample problem is of interest in itself relating to waves in nozzles with swirl and heat exchangers.

There are [1–3] three types of waves in a fluid in the absence of external restoring forces [4, 5], namely: (i) sound waves that are longitudinal and compressive; (ii) vortical waves that are transversal, hence incompressible; (iii) entropy modes associated with heat exchanges, hence non-isentropic flow.

The acoustic modes receive most attention because for an homogeneous uniform mean flow: (i) the acoustic modes satisfy the convected wave equation for uniform motion and the classical wave equation in a medium at rest [6–12]; (ii) by Kelvin circulation theorem the circulation along a loop convected with the mean flow is constant [13–17]; (iii) in homentropic conditions there are no entropy modes. The most general conditions for the existence of purely acoustic modes, decoupled from vortical-entropy modes, is a potential homentropic mean flow, that may be compressible, and leads to the high-speed wave equation [18–20] that reduces to the convected wave equation [21–23] in two cases: (i) uniform flow; (ii) low Mach number non-uniform flow. The presence of vorticity leads to acoustic-vortical-waves [24–29], in a compressible sheared [30–43] or swirling [44–51] mean flow. The present paper considers a further extension to acoustic-vortical-entropy waves that specify the stability of a compressible, vortical non-isentropic mean flow. The acoustic, vortical and entropy modes [1–3] are decoupled in a medium at rest and become coupled in sheared and/or swirling non-isentropic mean flows. The linearized Euler equations (LEE) contain all these modes, but consist of one vector (momentum) and three scalar (continuity, energy and state) equations with six variables (velocity vector, pressure, density and entropy). In this formulation, the ’wave operator’ is a 6×6 matrix that cannot be readily compared to a scalar wave equation for one variable like the pressure perturbation. This paper presents a scalar wave equation for a single wave variable (the radial velocity) that generalizes the classic wave equation for sound and the acoustic-vortical wave equation in a swirling flow. This derivation involves elimination among the 6 LEE equations for one variable only, namely the radial velocity, that determines through polarization relations all other variables, namely the perturbations of the density, pressure, temperature, entropy and axial and azimuthal components of the velocity. There is substantial evidence in the literature of the presence of non-acoustic perturbations in nozzle flows, and the derivation of an acoustic-vortical-entropy (AVE) equation aims to address this limitation of current wave equations, by allowing the interaction of all three effects.

The present paper: (i) is not about the generation of sound by small patches of vorticity [52, 53] or inhomogeneities [19, 20] convected in a potential flow, that is respectively ’vortex’ and ’entropy’ noise; (ii) it is about linear perturbations of a compressible, vortical, non-isentropic mean flow occupying all space, that may be designated acoustic-vortical-entropy waves. These perturbations determine the stability of the mean flow [54–60] in this case the stability of a compressible, vortical, non-isentropic flow. The paper considers what possibly is the simplest case of acoustic-vortical-entropy (AVE) waves: (i) linear non-dissipative perturbations of an axisymmetric mean flow with uniform axial velocity and rigid-body swirl; (ii) the mean flow is compressible, vortical and non-isentropic allowing for the existence of AVE waves; (iii) the perturbations depend on time, axial and radial coordinates, but not on azimuthal angle; (iv) this allows for the fundamental axisymmetric mode, but excludes all non-axisymmetric azimuthal modes. The derivation of the acoustic-vortical-entropy wave equation (Section 2): (i) is based on the linearization of the equations of continuity, inviscid momentum and energy (Subsection 2.2), using the entropy and equation of state of a perfect gas (Subsection 2.1); (ii) the elimination for the radial velocity perturbation leads to the AVE wave equation, and the remaining wave variables, namely the pressure, mass density, entropy, temperature and azimuthal velocity are expressed in terms of its solution (Subsection 2.3).

The presence of swirl leads to a radial pressure gradient in the mean flow due to the centrifugal force, and thus the sound speed varies radially; since the mean flow is assumed to be non-isentropic there is an entropy parameter, in addition to the sound speed. The acoustic-vortical-entropy wave equation specifying the radial dependence of the radial velocity perturbation spectrum for a given frequency and axial wavenumber has a singularity at a sonic radius (Section 3) where the swirl velocity of the mean flow equals the isothermal sound speed, i.e. the 'sonic condition' of isothermal swirl Mach number unity (Subsection 3.1). Thus there are two solutions: (i) an inner solution in ascending power series of the radius; (ii) an outer solution in descending power series of the radius.

2 The acoustic-vortical-entropy wave equation

The acoustic-vortical-entropy waves are considered as small axisymmetric perturbations of an axisymmetric compressible non-isentropic mean flow (Subsection 2.1) with uniform axial velocity and rigid body swirl (Subsection 2.2). Elimination for the radial velocity perturbation leads to the acoustic-vortical-entropy wave equation, whose solutions specifies also the perturbations of azimuthal velocity, pressure, mass density, temperature and entropy (Subsection 2.3).

2.1 Compressible, vortical, non-isentropic flow of a perfect gas

The fundamental equations of fluid mechanics are written in cylindrical coordinates (r, φ, z) in axisymmetric form without φ-dependence ($\partial/\partial\varphi = 0$):

(i) mass conservation:

$$\mathrm{D}\Gamma/\mathrm{d}t = -\Gamma\boldsymbol{\nabla}\cdot\boldsymbol{V} = -\frac{\Gamma}{r}\frac{\partial}{\partial r}(rV_r) - \Gamma\frac{\partial V_z}{\partial z}; \qquad (1)$$

(ii) inviscid momentum:

$$\Gamma\left(\mathrm{D}V_r/\mathrm{d}t - r^{-1}V_\varphi^2\right) + \partial_r P = 0, \qquad (2a)$$

$$\Gamma\left(\mathrm{D}V_\varphi/\mathrm{d}t + r^{-1}V_r V_\varphi\right) = 0, \qquad (2b)$$

$$\Gamma \mathrm{D}V_z/\mathrm{d}t + \partial_z P = 0; \qquad (2c)$$

(iii) energy neglecting dissipative effects, namely heat conduction and viscosity:

$$\Gamma T\, \mathrm{D}S/\mathrm{d}t = 0; \qquad (3)$$

(iv) state:

$$\mathrm{D}P/\mathrm{d}t = c^2 \mathrm{D}\Gamma/\mathrm{d}t + \beta \mathrm{D}S/\mathrm{d}t; \qquad (4)$$

where Γ is the mass density, P the pressure, V the velocity, T the temperature, S the entropy, the material derivative is denoted by

$$\mathrm{D}/\mathrm{d}t = \partial/\partial t + \boldsymbol{V}\cdot\boldsymbol{\nabla} = \partial/\partial t + V_r\partial_r + V_z\partial_z, \qquad (5a,b)$$

and the equation of state in the form (6a) specifies the coefficients in (4),

$$P = P(\Gamma, S): c^2 \equiv \left(\frac{\partial P}{\partial \Gamma}\right)_S, \beta = \left(\frac{\partial P}{\partial S}\right)_\Gamma, \qquad (6a\text{-}c)$$

namely the adiabatic sound speed (6b) and the non-isentropic coefficient (6c). Chemical reactions are not considered explicitly and appear through the entropy coefficient.

In the case of a perfect gas, the equations of state (7a) and entropy (7b),

$$P = R\Gamma T, \quad S = C_V \log P - C_P \log \Gamma, \qquad (7a,b)$$

involve the gas constant R and specific heats at constant volume C_V and pressure C_P that are related by (8a,c,d) involving the adiabatic exponent (8b),

$$R = C_P - C_V, \quad \gamma = \frac{C_P}{C_V}: \qquad (8a,b)$$

$$C_V = \frac{R}{\gamma - 1}, \quad C_P = \frac{\gamma R}{\gamma - 1}. \qquad (8c,d)$$

From the entropy equation (7b) it follows

$$\mathrm{d}S = C_V\frac{\mathrm{d}P}{P} - C_P\frac{\mathrm{d}\Gamma}{\Gamma}, \qquad (9a)$$

that the adiabatic sound speed (9b) is given by (9c),

$$\mathrm{d}S = 0 \,:\, c^2 = \left(\frac{\partial P}{\partial \Gamma}\right)_S = \frac{C_P}{C_V}\frac{P}{\Gamma} = \gamma\frac{P}{\Gamma} = \gamma RT \quad \text{9b,c)}$$

The non-isentropic coefficient (6c) may be calculated (10b) from the specific heat at constant volume (10a),

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_\Gamma \,: \qquad \text{(10a)}$$

$$\beta = \left(\frac{\partial P}{\partial T}\right)_\Gamma \Big/ \left(\frac{\partial S}{\partial T}\right)_\Gamma = \frac{T}{C_V}\left(\frac{\partial P}{\partial T}\right)_\Gamma \,; \qquad \text{(10b)}$$

in the case of a perfect gas (7a) follows (11a,b),

$$\beta = \frac{T}{C_V}R\Gamma = \frac{P}{C_V} = \frac{\gamma - 1}{R}P\,, \qquad \text{(11a-c)}$$

and also (11c) using (8c).

2.2 Linear perturbation of a uniform flow with rigid body swirl

The mean flow (subscript zero) is assumed to consist (12a) of a uniform axial velocity plus a rigid body swirl,

$$\boldsymbol{V}_0 = \boldsymbol{e}_z U + \boldsymbol{e}_\varphi \Omega r\,, \quad \boldsymbol{\varpi} = \boldsymbol{\nabla}\times\boldsymbol{V}_0 = \boldsymbol{e}_z 2\Omega\,, \quad \text{(12a,b)}$$

so that the vorticity (12b) is twice the angular velocity. The linearised material derivative (5a) for the axial mean flow (12a) is (13a):

$$\mathrm{d/dt} \equiv \partial/\partial t + U\partial/\partial z\,, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{V}_0 = 0\,, \quad \text{(13a,b)}$$

and the mean flow velocity (12a) has zero divergence (13b). Applying the fundamental equations to the mean flow (12a) it follows that: (i-ii) the mass density (1) and entropy (3) can depend only on the radius (14a,b); (iii) there is a radial pressure gradient (2a) due to the centrifugal force (14c),

$$\rho_0 = \rho_0(r)\,,\; s_0 = s_0(r) \,:\, p_0' \equiv \mathrm{d}p_0/\mathrm{d}r = \rho_0\Omega^2 r\,; \quad \text{(14a-c)}$$

(iv) the mean flow is incompressible (13b) and the assumption of a constant mass density (15a) leads to the pressure (15c) where (15b) is the pressure on axis,

$$\rho_0 = \mathrm{const}\,,\; p_{00} = p_0(0) \,:\, p_0(r) = p_{00} + \frac{1}{2}\rho_0\Omega^2 r^2\,. \quad \text{(15a-c)}$$

The sound speed (9c) and non-isentropic coeficient (11b) are given in the mean flow respectively by (16b) and (16c), where (16a) is the sound speed on the axis,

$$c_{00}^2 = \gamma\frac{p_{00}}{\rho_0} \,: \qquad (16\mathrm{a-c})$$

$$[c_0(r)]^2 = \gamma\frac{p_0(r)}{\rho_0} = c_{00}^2 + \frac{\gamma}{2}\Omega^2 r^2\,,\; \beta_0(r) = \frac{p_0(r)}{C_V} \qquad (16\mathrm{a-c})$$

The entropy in the mean flow (17a),

$$s_0 = C_V \log p_0 - C_P \log \rho_0\,, \qquad \text{(17a)}$$

has radial gradient (17b),

$$s_0' = C_V\frac{p_0'}{p_0} = C_V\frac{\rho_0\Omega^2 r}{p_0} = C_V\gamma\frac{\Omega^2 r}{c_0^2} = C_P\frac{\Omega^2 r}{c_0^2}\,. \qquad \text{(17b)}$$

Thus the uniform axial flow with rigid body swirl (12a) and a constant mass density (15a) implies the radial dependences of the pressure (15b,c), sound speed (16a,b) and also the existence of an entropy gradient (17b). The linear perturbation of this mean flow is considered next.

The total flow is assumed to consist of the mean flow plus a perturbation depending on time *t*, radial *r* and axial *z* coordinate, but not on the azimuthal coordinate *φ*,

$$V_r(r,z,t) = v_r(r,z,t)\,, \qquad (18\mathrm{a})$$

$$V_\varphi(r,z,t) = \Omega r + v_\varphi(r,z,t)\,, \qquad (18\mathrm{b})$$

$$V_z(r,z,t) = U + v_z(r,z,t)\,, \qquad (18\mathrm{c})$$

$$P(r,z,t) = p_0(r) + p(r,z,t)\,, \qquad (18\mathrm{d})$$

$$\Gamma(r,z,t) = \rho_0 + \rho(r,z,t)\,, \qquad (18\mathrm{e})$$

$$S(r,z,t) = s_0(r) + s(r,z,t)\,. \qquad (18\mathrm{f})$$

Since the mean flow properties, that appear as coeficients in the linearisation, depend on *r* but not *(z; t)*, the Fourier transform is made (19) with frequency *ω* and axial wavenumber *k*,

$$f(r,z,t) = \int_{-\infty}^{+\infty}\mathrm{d}k\int_{-\infty}^{+\infty}\mathrm{d}\omega\, e^{\mathrm{i}(kx-\omega t)}\tilde{f}(r,k,\omega)\,; \qquad (19)$$

for example the linearised material derivative for the axial flow (13a) leads (20a) to the frequency (20b) Doppler shifted by the axial mean flow,

$$\mathrm{d/dt} \to -\mathrm{i}\omega_* \,: \quad \omega_* = \omega - kU\,. \qquad \text{(20a,b)}$$

Substituting (18a{f) in (1,2a{c,3,4) and linearising leads to

$$\mathrm{i}\omega_* r\tilde{\rho} - \rho_0 (r\tilde{v}_r)' - \mathrm{i}\rho_0 k r \tilde{v}_z = 0\,, \quad (21a)$$

$$\mathrm{i}\rho_0\omega_*\tilde{v}_r + 2\Omega\rho_0\tilde{v}_\varphi + \Omega^2 r\tilde{\rho} - \tilde{p}' = 0 \quad (21b)$$

$$\mathrm{i}\omega_*\tilde{v}_\varphi - 2\Omega\tilde{v}_r = 0\,, \quad (21c)$$

$$\rho_0\omega_*\tilde{v}_z - k\tilde{p} = 0\,, \quad (21d)$$

$$\mathrm{i}\omega_*\tilde{s} = s_0'\tilde{v}_r = C_P\frac{\Omega^2}{c_0^2} r\tilde{v}_r\,, \quad (21e)$$

$$\tilde{p} = c_0^2\tilde{\rho} + \beta_0\tilde{s}\,. \quad (21f)$$

The last equation (21f) follows from linearization of (4),

$$\mathrm{i}\omega_*(\tilde{p} - c_0^2\tilde{\rho} - \beta_0\tilde{s}) = \tilde{v}_r(p_0' - c_0^2\rho_0' - \beta_0 s_0') = 0\,, \quad (22)$$

using (15a) and (17b). The energy equation (3) simplifies to (23a) for a perfect gas (7a),

$$\frac{P}{R}\frac{DS}{dt} = 0 : \quad p_0\frac{\mathrm{d}s_0}{\mathrm{d}t} = 0\,, \quad (23a,b)$$

$$0 = (p + p_0)\frac{\mathrm{D}(s + s_0)}{\mathrm{d}t} - p_0\frac{\mathrm{d}s_0}{\mathrm{d}t}\,, \quad (23c)$$

implying that: (i) the mean flow is isentropic (23b), that is consistent (13) with the entropy being a function of the radius (14b); (ii) subtracting the mean state (23b) from the exact energy equation (23a) leads to (23c) that is linearised (23d),

$$0 = p_0\frac{\mathrm{d}s}{\mathrm{d}t} + p_0\left(\boldsymbol{V}\cdot\boldsymbol{\nabla}s_0\right)\,; \quad (23d)$$

(iii) from (23d) follows (23e),

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\left(\boldsymbol{V}\cdot\boldsymbol{\nabla}s_0\right)\,, \quad \mathrm{i}\omega_*\tilde{s} = s_0'\tilde{v}_r\,, \quad (23e,f)$$

proving (23f)≡(21e).

2.3 Wave equation for the radial velocity and polarization relations

Of the six variables in (21a-f) four ($\tilde{v}_z$; $\tilde{v}_\varphi$; $\tilde{\rho}$;$\tilde{s}$) are expressible (21d,c,e,a) in terms of ($\tilde{p}$;$\tilde{v}_r$),

$$\tilde{v}_z = \frac{k}{\rho_0\omega_*}\tilde{p}\,, \quad \tilde{v}_\varphi = -\mathrm{i}\frac{2\Omega}{\omega_*}\tilde{v}_r\,, \quad \tilde{s} = -\mathrm{i}C_P\frac{\Omega^2}{c_0^2\omega_*}r\tilde{v}_r\,. \quad (24a-c)$$

$$\tilde{\rho} = -\mathrm{i}\frac{\rho_0}{\omega_* r}(r\tilde{v}_r)' + \frac{k^2}{\omega_*^2}\tilde{p}\,. \quad (24d)$$

Substituting (24c,d) in (21f) leads to

$$\mathrm{i}\tilde{p}\left(\omega_* - k^2c_0^2/\omega_*\right) = \rho_0\left(\Omega^2 r + c_0^2/r\right)\tilde{v}_r + \rho_0 c_0^2\tilde{v}_r'\,, \quad (25)$$

the pressure in terms of the radial velocity spectrum. Substituting (24b,d) in (21b) leads to a relation between $\tilde{p}$ and $\tilde{v}_r$ distinct from (25), namely

$$\mathrm{i}\rho_0\left[\left(\omega_*^2 - 5\Omega^2\right)\tilde{v}_r - \Omega^2 r\tilde{v}_r'\right] = \omega_*\tilde{p}' - \frac{k^2\Omega^2 r}{\omega_*}\tilde{p}\,. \quad (26)$$

Substituting $\tilde{p}$ from (25) in (26) leads to the acoustic-vortical-entropy wave equation for the radial velocity perturbation spectrum,

$$c_0^2\tilde{v}_r'' + A\tilde{v}_r' + B\tilde{v}_r = 0\,, \quad (27)$$

with coefficients

$$X \equiv 1 - k^2c_0^2/\omega_*^2 : \quad A = c_0^2/r + X\left[c_0^2/X\right]'\,, \quad (28a,b)$$

$$B = \left(\omega_*^2 - 5\Omega^2\right)X - k^2\Omega^2\left(\Omega^2 r^2 + c_0^2\right)/\omega_*^2 + X\left[\left(\Omega^2 r + c_0^2/r\right)/X\right] \quad (28c)$$

In conclusion the axisymmetric compressive, vortical, non-isentropic perturbations of a uniform axial flow with rigid body swirl (12a), with frequency ω and axial wavenumber k, lead (19) to the acoustic-vortical entropy wave equation (27) with coefficients (28a-c) satisfied by the radial velocity perturbation spectrum. The other wave variables are specified by the following polarization relations: (i-iii) the pressure (25), entropy (24c) and azimuthal velocity (24b) perturbation spectra; (iv-v) the axial velocity (24a) and mass density (24d) perturbation spectra lead, by (25), respectively to (29a) and (29b),

$$\tilde{v}_z = -\mathrm{i}k\left[\left(\Omega^2 r + c_0^2/r\right)\tilde{v}_r + c_0^2\tilde{v}_r'\right]/\left(\omega_*^2 - k^2c_0^2\right)\,, \quad (29a)$$

$$\mathrm{i}\tilde{\rho}/\rho_0 = \tilde{v}_r'/\omega_* + \tilde{v}_r/(\omega_* r) + k^2\left[\left(\Omega r + c_0^2/r\right)\tilde{v}_r + c_0^2\tilde{v}_r'\right]/\left(\omega_*^3 - k^2c_0^2\omega_*\right)$$

The temperature perturbation spectrum follows from the equation of state (7a),

$$R\tilde{T} = \frac{\mathrm{i}c_0^2}{\omega_*\gamma}\left(\tilde{v}_r' + \tilde{v}_r/r\right) - \mathrm{i}\frac{\omega_* - k^2c_0^2/\gamma\omega_*}{\omega_*^2 - k^2c_0^2}\left[\left(\Omega^2 r + c_0^2/r\right)\tilde{v}_r + c_0^2\tilde{v}_r'\right]\,, \quad (30a,b)$$

using (29b) and (25).

3 Monotonic and oscillatory inner and outer solutions

The acoustic-vortical-entropy wave equation with zero axial wavenumber is solved exactly as ascending (Subsection 3.2) and descending (Subsection 3.3) power series of the radius that converge respectively inside and outside a sonic radius, where the isothermal swirl Mach number is unity. This specifies the separation condition between oscillatory and monotonic dependence of the radius of the AVE wave perturbation of the compressible, vortical, non-isentropic mean flow: (i) near the axis oscillatory solutions correspond to the frequency larger than the vorticity (Subsection 3.1); (ii) at infinity the condition for oscillatory solutions is opposite, that is the vorticity must exceed the frequency (Subsection 3.3).

3.1 Condition separating oscillatory from monotonic radial dependences

If the axial wavenumber is not zero, the vanishing of (28a) introduces singularities in the AVE wave equation (27). The condition *X=0* corresponding to $\pm kc_0=\omega_*=\omega-kU$ leads to a singularity of the wave equation similar to those that occur for acoustic-shear [30-43] and acoustic-vortical [44-51] waves and may be addressed in future work. The present paper concentrates on non-isentropic eects, in the simpler case of zero axial wavenumber (31a), that is neglecting axial dependence, when there is (20b) no Doppler shift (31b) and the coefficients of the wave equation (28a-c) simplify respectively to (31d-f),

$$k=0\,,\;\omega_*=\omega\,,\;(c_0^2)'=\gamma\Omega^2 r\,,X=1\;: \quad (31\mathrm{a-d})$$

$$A=c_0^2/r+(c_0^2)'=\gamma\Omega^2 r+c_0^2/r\,, \quad (31\mathrm{e})$$

$$B=\omega^2-4\Omega^2+\gamma\Omega^2-c_0^2/r^2\,, \quad (31\mathrm{f})$$

where the radial dependence of the sound speed (16c) was used (31c). Thus the acoustic-vortical-entropy wave equation (27) for (31a-f) an axisymmetric mode of frequency ω simplifies to

$$c_0^2\tilde{v}_r''+\left(\gamma\Omega^2 r+c_0^2/r\right)\tilde{v}_r'+\left[\omega^2+(\gamma-4)\Omega^2-c_0^2/r^2\right]\tilde{v}_r=0\,. \quad (32)$$

The radial dependence of the sound speed (16b) is quadratic (33a),

$$[c_0(r)]^2=c_{00}^2\left[1+(r/r_0)^2\right]\;,\;r_0=(c_{00}/\Omega)\sqrt{2/\gamma}\,, \quad (33\mathrm{a,b})$$

with reference radius (33b). Substituting (33b) in the wave equation (32) leads to

$$r^2\left(1+r^2/r_0^2\right)\tilde{v}_r''+r\left(1+3r^2/r_0^2\right)\tilde{v}_r'+ \\ +\left\{\left[(\omega/c_{00})^2+(1-8/\gamma)/r_0^2\right]r^2-1\right\}\tilde{v}_r=0\,. \quad (34)$$

Using (31a) and (33a,b), the remaining wave variables are the azimuthal velocity (24b), mass density (29b), temperature (30a), entropy (24c) and pressure (25) specified respectively by (35a-e),

$$\tilde{v}_\varphi=-\mathrm{i}\frac{2\Omega}{\omega}\tilde{v}_r\,,\quad \tilde{\rho}=-\mathrm{i}(\rho_0/\omega)\left(\tilde{v}_r'+\tilde{v}_r/r\right)\,, \quad (35\mathrm{a,b})$$

$$\tilde{T}/T_0=\left[(\gamma/c_0^2)\tilde{p}-\tilde{\rho}\right]/\rho_0\,,\;\tilde{s}=-\mathrm{i}\frac{2}{\omega}\frac{C_V r\tilde{v}_r}{r^2+r_0^2}\,, \quad (35\mathrm{c,d})$$

$$\tilde{p}=-\mathrm{i}\frac{\rho_0\gamma\Omega^2}{2\omega}\left[\left(r+\frac{2r}{\gamma}+\frac{r_0^2}{r}\right)\tilde{v}_r+\left(r^2+r_0^2\right)\tilde{v}_r'\right. \quad (35\mathrm{e})$$

in terms of the radial velocity perturbation spectrum.

The adiabatic exponent for a perfect gas is given by (36b) where (36a) is the number of degrees of freedom of a molecule,

$$N=3,5,6\;:\quad \gamma=1+\frac{2}{N}=\frac{5}{3},\frac{7}{5},\frac{4}{3}, \quad (36\mathrm{a,b})$$

namely: (i) three for monoatomic gas; (ii) five for a diatomic gas or polyatomic gas with molecules in a line; (iii) six for a three dimensional polyatomic molecule. The reference radius (33b) corresponds to a ratio of the azimuthal velocity of the mean flow to the sound speed on axis given by

$$\frac{r_0\Omega}{c_{00}}=\sqrt{\frac{2}{\gamma}}=\sqrt{\frac{2N}{N+2}}=\sqrt{\frac{6}{5}},\sqrt{\frac{10}{7}},\sqrt{\frac{3}{2}}, \quad (37)$$

that is of order unity and plays the role of swirl Mach number at the axis, bearing in mind that the sound speed (33a,b) is not constant. Using the sound speed (33a) at the sonic radius (38a) leads to (38b),

$$c_0(r_0)=c_{00}\sqrt{2}\;:\quad r_0\Omega=\frac{c_0(r_0)}{\sqrt{\gamma}}=\sqrt{RT_0(r_0)}=\bar{c}_0(r_0)\,, \quad (38\mathrm{a,b})$$

showing that the sonic radius corresponds to azimuthal velocity equal to the isothermal sound speed, that is isothermal swirl Mach number unity. Since vortical modes are transversal and hence incompressible, the relevant sound speed and Mach number are isothermal. If the radius is small compared with the reference radius (39a), that is for small swirl isothermal Mach number, the wave equation (34) simplifies to (39b),

$$r^2 \ll r_0^2: \quad r^2\tilde{v}_r'' + r\tilde{v}_r' + \left(\chi^2 r^2 - 1\right)\tilde{v}_r = 0, \qquad \text{(39a,b)}$$

in the passage from (32) to (39b) the approximation (39a) was made in the coefficient of $\widetilde{v_{0r}}$, but not in the coefficient of $\widetilde{v_r}$, because the frequency (40c) could be large (40b) in the radial wavenumber (40a),

$$\chi \equiv \kappa/r_0, \quad \kappa^2 = \varpi^2 + 1 - 8/\gamma, \quad \varpi \equiv \omega r_0/c_{00}. \qquad \text{(40a-c)}$$

Thus the approximation of small radius (39a) leads to the Bessel equation (39b) where (40b) is the dimensionless radial wavenumber involving the dimensionless frequency (40c).

The Bessel equation has oscillatory solutions for real wavenumber and monotonic increasing solutions for imaginary wavenumber. Although the preceding result was obtained only for small radius (39a), it will be extended in the sequel (Subsections 3.2 and 3.3) to all values of the radial distance. Thus the condition specifying wave fields with oscillatory dependence on the radius (41a) is expressed in terms of the dimensionless frequency (40b),

$$\kappa^2 > 0: \quad \frac{\omega r_0}{c_{00}} > \sqrt{\frac{8}{\gamma} - 1} = \sqrt{\frac{7N-2}{N+2}}. \qquad \text{(41a, b)}$$

Using (33b) the condition for radially oscillatory AVE waves is written in terms of the angular velocity,

$$\omega > \frac{c_{00}}{r_0}\sqrt{\frac{8}{\gamma} - 1} = \Omega\sqrt{\frac{\gamma}{2}\left(\frac{8}{\gamma} - 1\right)} = \Omega\sqrt{\frac{7}{2} - \frac{1}{N}}. \qquad \text{(42)}$$

Using the sound speed (38a) at the reference radius the oscillatory condition (41b) becomes

$$\frac{\omega r_0}{c_0(r_0)} = \frac{\omega r_0}{c_{00}\sqrt{2}} > \sqrt{\frac{4}{\gamma} - \frac{1}{2}} = \sqrt{\frac{7N-2}{2N+4}}. \qquad \text{(43)}$$

Bearing in mind that the modulus of the vorticity is twice the angular velocity (12b) the oscillatory condition (42) becomes

$$\frac{\omega}{|\varpi|} = \frac{\omega}{2\Omega} > \sqrt{1 - \frac{\gamma}{8}} = \sqrt{\frac{7}{8} - \frac{1}{4N}} \equiv \mu. \qquad \text{(44)}$$

Of the four forms of the oscillatory condition (41b), (42), (43) and (44) the last is independent of the geometry and may be the most general: a compressible, vortical, nonisentropic flow has perturbations with oscillatory dependence on the radial distance if the frequency is larger than the maximum of the modulus of the peak vorticity ϖ multiplied by the factor μ in (44). The spatial growth of perturbations of acoustic-vortical waves [49, 50] is comparable to the temporal growth [51] as an indicator of instability. Thus the oscillatory condition excluding monotonic growth of perturbations could be equivalent to a stability condition for the mean flow. This conjecture can be applied (Figure 1) to combustion stability in a confined space: (i) if the natural frequencies exceed the product $\mu\varpi$ there is (Figure 1a) stability, and only the fundamental frequency needs to be considered $\omega_1 > \mu\varpi$; (ii) if the fundamental frequency and other modes lie below $\mu\varpi$ those modes lead to instability (Figure 1b).

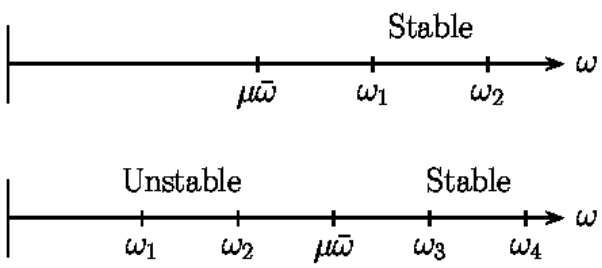

Fig.1. The compressible, vortical, non-isentropic flow is stable if the peak vorticity multiplied by (44) is less than the fundamental frequency (Figure 1a) and unstable otherwise (Figure 1b).

The passage from stable to the unstable case could be due to: (i) increasing the vorticity of the mean flow, e.g. to achieve better mixing for 'lean' fuel saving combustion; (ii) increasing the size of the enclosure, so that the natural frequencies reduce, and fall below $\mu\varpi$. The remark (i) agrees with the observation that lean combustion tends to be unstable; the remark (ii) agrees with the observation that larger rocket motors are more prone to large amplitude oscillations. The stability criterion

$$\omega_1 > \mu\varpi_{max}, \mu = 0{:}890; 0{:}908; 0{:}913, \qquad \text{(40a-c)}$$

that the fundamental frequency must be larger than the modulus of the peak vorticity times the factor (44) can be tested for more complex geometries using numerical codes. Similar conditions were obtained before for the stability of an inviscid boundary layer [56, 58] and for sound in vertical flows [59]. It has a simple interpretation: (i) acoustic modes with frequency ω are stable; (ii) vortical modes with vorticity ϖ are unstable; (iii) there is stability if the acoustic modes predominate $\omega>|\varpi|$; (iv) there is instability if the vortical modes predominate $|\varpi|>\omega$. The factor (44) involving the adiabatic exponent appears because the vertical modes are incompressible and the acoustic modes are adiabatic and thus the ratio of frequency to vorticity is close to but not exactly unity. The stability condition (45a,b) was established from the AVE wave equation (39b) for small radius (39a). It can be shown that the equivalent condition for AVE waves with oscillatory radial dependence is not restricted to small radius (39a) and applies to any radial distance smaller than the sonic radius.

4 Further studies

The AVE wave equation can be transformed to a Gaussian hypergeometric differential equation thus confirming the inner and outer solution as respectively ascending and descending power series of the radius, valid respectively inside and outside the sonic radius. The inner and outer solutions are matched by using a third solution valid around the sonic radius that overlaps with both; this third solution is valid over the whole space and shows that the wave field is finite at the sonic radius.

The numerical solutions of the AVE wave equation resulted in the computation of the wave variables. Thus the divergence of the inner and outer solutions at the sonic radius is due to the failure of the power series to converge at their boundary of convergence and not to the wave field that is finite at the sonic radius. The inner and outer solutions may be used to describe the wave field respectively near the axis and asymptotically for large radius; they apply to waves in a cylinder or cylindrical annulus, respectively inside and outside the sonic radius; the solutions around the sonic radius still apply also when the sonic radius lies inside the cylindrical or annular duct. The solutions of the AVE wave equation that hold for all finite non-zero values of the radius are applied to a cylinder with rigid walls containing the sonic radius to determine: (i) the eigenvalues for the radial wavenumber and frequency; (ii) the corresponding eigenfunctions or waveforms for the perturbations of the radial and azimuthal velocity, mass density, entropy, pressure and temperature as function of the radius.

5 Conclusions

The present paper may be the first to derive a scalar wave equation with a single variable combining the interactions of the three types of waves in a fluid not subject to external force fields, hence the designation acoustic-vortical-entropy (AVE) waves. A deliberate choice was made of one of the simplest baseline flows that could support AVE waves, namely an incompressible nonisentropic uniform flow with rigid body swirl, leading to a mean flow pressure and sound speed varying radially due to the centrifugal force. The linear non-dissipative perturbation of this mean flow leads in the axisymmetric case to the AVE wave equations (27;28a-c) first obtained here. The exact solution is obtained in terms of Gaussian hypergeometric functions in the case of zero axial wavenumber, when there are only temporal and radial dependences. The six wave variables in this case are the frequency spectra of the perturbations of the (i) radial and (ii) azimuthal velocity, (iii) mass density, (iv) entropy, (v) pressure and (vi) temperature.

Acknowledgements

This work was supported by FCT (Foundation for Science and Technology) through IDMEC (Institute of Mechanical Engineering), under LAETA, project Pest-OE/EME/LA0022.

References

[1] L. S. G. Kovasznay, Turbulence in supersonicow, J Aero Sci 20 (10) (1953)

[2] L. D. Landau, E. M. Lifshitz, Fluid Mechanics, Vol. 6 of Course of Theoretical Physics, Pergamon Press, 1959, ISBN:9780080091044.

[3] A. D. Pierce, Acoustics: An Introduction to Its Physical Principles and Applications, McGraw-Hill, 1981, ISBN:9780883186121.

[4] L. M. B. C. Campos, On the generation and radiation of magneto-acoustic waves, J Fluid Mech 81 (3) (1977) 529-549, doi:10.1017/S0022112077002213.

[5] L. M. B. C. Campos, On magnetoacoustic-gravity-inertial (MAGI) waves I. generation, propagation, dissipation and radiation, Mon Not R Astron Soc 410 (2) (2011) 717-734, doi:10.1111/j.1365-2966.2010.17553.x.

[6] J. W. S. Rayleigh, The Theory of Sound, 2nd Edition, Vol. 2 Vols. of Dover Books on Physics, Dover Publications, 1945 (reprint), 1877, ISBN:9780486602929

[7] P. M. Morse, K. U. Ingard, Theoretical Acoustics, McGraw-Hill, 1968, ISBN:0691084254.

[8] M. E. Goldstein, Aeroacoustics, McGraw-Hill, 1976, ISBN:9780070236851.

[9] J. Lighthill, Waves in Fluids, Cambridge Mathematical Library, Cambridge U.P., 1978, ISBN:9780521010450.

[10] A. P. Dowling, J. E. Ffowcs-Williams, Sound and Sources of Sound, Ellis Horwood, 1983, ISBN:9780853125273.

[11] T. D. Rossing (Ed.), Handbook of Acoustics, Springer handbooks, Springer, 2007, ISBN:9780387336336.

[12] M. J. Crocker (Ed.), Handbook of Noise and Vibration Control, Wiley, 2007, ISBN:9780471395997.

[13] M. J. Crocker (Ed.), Handbook of Noise and Vibration Control, Wiley, 2007, ISBN:9780471395997.

[14] L. M. Milne-Thomson, Theoretical Hydrodynamics, Dover, 1968, ISBN:0486689700.

[15] J. Lighthill, An Informal Introduction to Theoretical Fluid Mechanics, Institute of Mathematics & its Applications Monograph Series, Cambridge U.P., 1988, ISBN:9780198536307.

[16] M. S. Howe, Hydrodynamics and Sound, Cambridge U.P., 2006, ISBN:9780521868624.

[17] L. M. B. C. Campos, Complex Analysis with Applications to Flows and Fields, Vol. 1 of Mathematics and Physics in Science and Engineering, CRC Press, 2010, ISBN:9781420071184.

[18] M. S. Howe, Contributions to the theory of aerodynamic sound, with application to excess jet noise and the theory of the flute, J Fluid Mech 71 (5) (1975) 625-673, doi:10.1017/S0022112075002777.

[19] L. M. B. C. Campos, On the emission of sound by an ionized inhomogeneity, P Roy Soc Lond A Mat 359 (1696) (1978) 65-91, doi:10.1098/rspa.1978.0032.

[20] L. M. B. C. Campos, F. J. P. Lau, On sound generation by moving surfaces and convected sources in a flow, Int J Aeroacoust 11 (1) (2012) 103-136, doi:10.1260/1475-472X.11.1.103.

[21] L. M. B. C. Campos, On linear and non-linear wave equations for the acoustics of high-speed potential flows, J Sound Vib 110 (1) (1986) 41-57,doi:10.1016/S0022-460X(86)80072-4.

[22] L. M. B. C. Campos, On waves in gases.part I: Acoustics of jets, turbulence, and ducts, Rev Mod Phys 58 (1) (1986) 117-182, doi:10.1103/RevModPhys.58.117.

[23] L. M. B. C. Campos, On the generalizations of the doppler factor, local frequency, wave invariant and group velocity, Wave Motion 10 (3) (1988) 193-207, doi:10.1016/0165-2125(88)90018-2.

[24] W. Haurwitz, Zur theorie der wellenbewegungen in luft und wasser ("on the theory of wave perturbations in a flow in air and water"), Veroentlichungen des Geophysikalischen Instituts der Karl-Marx-Universit□at Leipzig 6 (1) (1931) 334-364.

[25] D. Kuchemann, St□orungsbewegungen in einer gasstr□omung mit grenzschicht, Z Angew Math Mech 18 (4) (1938) 207-222, doi:10.1002/zamm.19380180402.

[26] D. C. Pridmore-Brown, Sound propagation in a fluid flowing through an attenuating duct, J Fluid Mech 4 (4) (1958) 393-406, doi:10.1017/S0022112058000537.

[27] W. Mohring, E. Muller, F. Obermeier, Problems in flow acoustics, Rev Mod Phys 55 (3) (1983) 707-724, doi:10.1103/RevModPhys.55.707.

[28] L. M. B. C. Campos, On 36 forms of the acoustic wave equation in potential flows and inhomogeneous media, Appl Mech Rev 60 (4) (2007) 149-171, doi:10.1115/1.2750670.

[29] L. M. B. C. Campos, On 24 forms of the acoustic wave equation in vertical flows and dissipative media, Appl Mech Rev 60 (6) (2007) 291-315, doi:10.1115/1.2804329.

[30] M. Goldstein, E. Rice, Effect of shear on duct wall impedance, J Sound Vib 30 (1) (1973) 79{84, doi:10.1016/S0022-460X(73)80051-3.

[31] D. S. Jones, The scattering of sound by a simple shear layer, Phil Trans R Soc Lond A 284 (1323) (1977) 287-328, doi:10.1098/rsta.1977.0011.

[32] D. S. Jones, Acoustics of a splitter plate, IMA J Appl Math 21 (2) (1978) 197-209, doi:10.1093/imamat/21.2.197.

[33] S. P. Koutsoyannis, Characterization of acoustic disturbances in linearly sheared flows, J Sound Vib 68 (2) (1980) 187-202, doi:10.1016/0022-460X(80)90464-2.

[34] S. P. Koutsoyannis, K. Karamcheti, D. C. Galant, Acoustic resonances and sound scattering by a shear layer, AIAA J 18 (12) (1980) 1446-1454, doi:10.2514/3.7736.
[35] J. N. Scott, Propagation of sound waves through a linear shear layer, AIAA J 17 (3) (1979) 237-244, doi:10.2514/3.61107.
[36] L. M. B. C. Campos, J. M. G. S. Oliveira, M. H. Kobayashi, On sound propagation in a linear shear ow, J Sound Vib 219 (5) (1999) 739-770, doi:10.1006/jsvi.1998.1880.
[37] L. M. B. C. Campos, P. G. T. A. Serrao, On the acoustics of an exponential boundary layer, Phil Trans R Soc Lond A 356 (1746) (1998) 2335-2378, doi:10.1098/rsta.1998.0277.
[38] L. M. B. C. Campos, M. H. Kobayashi, On the re ection and transmission of sound in a thick shear layer, J Fluid Mech 424 (2000) 303-326, doi:10.1017/S0022112000002068.
[39] L. M. B. C. Campos, J. M. G. S. Oliveira, On the acoustic modes in a duct containing a parabolic shear flow, J Sound Vib 330 (6) (2011) 1166-1195, doi:10.1016/j.jsv.2010.09.021.
[40] L. M. B. C. Campos, M. H. Kobayashi, On the propagation of sound in a highspeed non-isothermal shear flow, Int J Aeroacoust 8 (3) (2009) 199-230, doi:10.1260/147547208786940035.
[41] L. M. B. C. Campos, M. H. Kobayashi, Sound transmission from a source outside a nonisothermal boundary layer, AIAA J 48 (5) (2010) 878-892, doi:10.2514/1.40674.
[42] L. M. B. C. Campos, M. H. Kobayashi, On an acoustic oscillation energy for shear flows, Int J Aeroacoust 12 (1) (2013) 123-168, doi:10.1260/1475-472X.12.1-2.123.
[43] L. M. B. C. Campos, M. H. Kobayashi, On sound emission by sources in a shear flow, Int J Aeroacoust 12 (7-8) (2013) 719-742, doi:10.1260/1475-472X.12.7-8.719.
[44] V. V. Gobulev, H. M. Atassi, Sound propagation in an annular duct with mean potential swirling flow, J Sound Vib 198 (5) (1996) 601-616, doi:10.1006/jsvi.1996.0591.
[45] V. V. Gobulev, H. M. Atassi, Acousticvorticy waves in swirling flows, J Sound Vib 209 (2) (1998) 203-222, doi:10.1006/jsvi.1997.1049.
[46] L. M. B. C. Campos, P. G. T. A. Serrao, On the sound in unbounded and ducted vortex flows, SIAM J Appl Math 65 (4) (2005) 1353-1368, doi:10.1137/S0036139903427076.
[47] C. K. W. Tam, L. Auriault, The wave modes in ducted swirling flows, J Fluid Mech 371 (1) (1998) 1-20, doi:10.1017/S0022112098002043.
[48] M. E. Goldstein, Unsteady vertical and entropic distortions of potential flows round arbitrary obstacles, J Fluid Mech 89 (3) (1978) 433-468, doi:10.1017/S0022112078002682.
[49] C. J. Heaton, N. Peake, Acoustic scattering in a duct with mean swirling flow, J Fluid Mech 540 (2005) 189-220, doi:10.1017/S0022112005005719.
[50] C. J. Heaton, N. Peake, Algebraic and exponential instability of inviscid swirling flow, J Fluid Mech 565 (2006) 279-318, doi:10.1017/S0022112006001698.
[51] L. M. B. C. Campos, P. G. T. A. Serrao, On the continuous and discrete spectrum of acoustic-vortical waves, Int J Aeroacoust 12 (7-8) (2013) 743-782, doi:10.1260/1475-472X.12.7-8.743.
[52] A. Powell, Theory of vortex sound, J Acoust Soc Am 36 (1) (1964) 177-195, doi:10.1121/1.1918931.
[53] M. S. Howe, Theory of vortex sound, Cambridge Texts in Applied Mathematics, Cambridge U.P., 2002, ISBN:9780521012232.
[54] C. C. Lin, The Theory of Hydrodynamic Stability, Cambridge Monographs on Mechanics Applied Mathematics, Cambridge U.P., 1955, aSIN:B0000CJB1L.
[55] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford U.P., 1961, ISBN:978-0486640716.
[56] A. Michalke, On spatially growing disturbances in an inviscid shear layer, J Fluid Mech 23 (3) (1965) 521-544, doi:10.1017/S0022112065001520.
[57] D. D. Joseph, Stability of fluid motions. I,II, Vol. 27 & 28 of Springer Tracts in Natural Philosophy, Springer-Verlag, 1976, ISBN:9780471116219.
[58] A. Michalke, Instability of a compressible circular free jet with consideration of the influence of the jet boundary layer thickness, Technical Memorandum TM 75190, NASA (1977).
[59] S. E. P. Bergliaa, K. Hibberd, M. Stone, M. Visser, Wave equation for sound in fluids with vorticity, Physica D 191 (1-2) (2004) 121-136, doi:10.1016/j.physd.2003.11.007.
[60] L. M. B. C. Campos, Transcendental Representations with applications to Solids and Fluids, Vol. 2 of Mathematics and Physics in Science and Engineering, CRC Press, 2012, ISBN:9781439834312.

Copyright Statement

The authors confirm that they, and/or their company or organization, hold copyright on all of the original material included in this paper. The authors also confirm that they have obtained permission, from the copyright holder of any third party material included in this paper, to publish it as part of their paper. The authors confirm that they give permission, or have obtained permission from the copyright holder of this paper, for the publication and distribution of this paper as part of the ICAS 2016 proceedings or as individual off-prints from the proceedings.